Counting the Solutions to a Query

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- Three fundamental associated problems: enumeration, uniform generation and counting of solutions
- Study these three problems together
  - In line with [JVV86]
- Define classes (of queries) that have good properties in terms of these three problems

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- The motivating scenario comes from graph databases: counting paths conforming to a regular expression

In this talk

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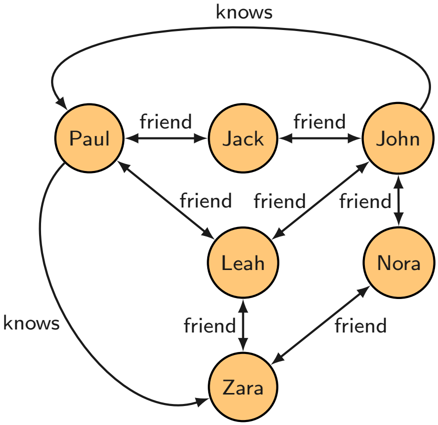
  - The motivating scenario comes from relational databases: counting the number of answers to an acyclic conjunctive query

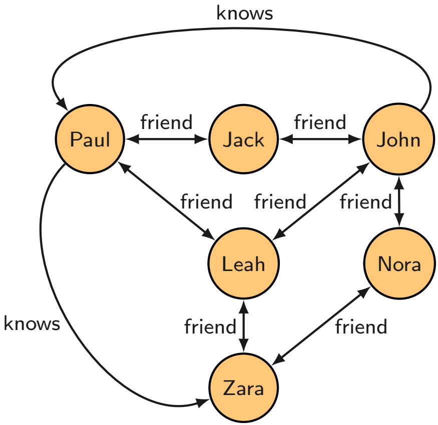
Definition of the setting

Our first motivating scenario: graph databases

G :



A query over G: (friend + knows)*



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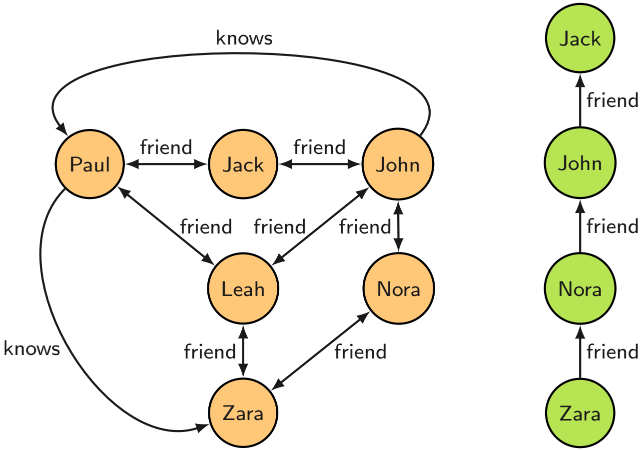

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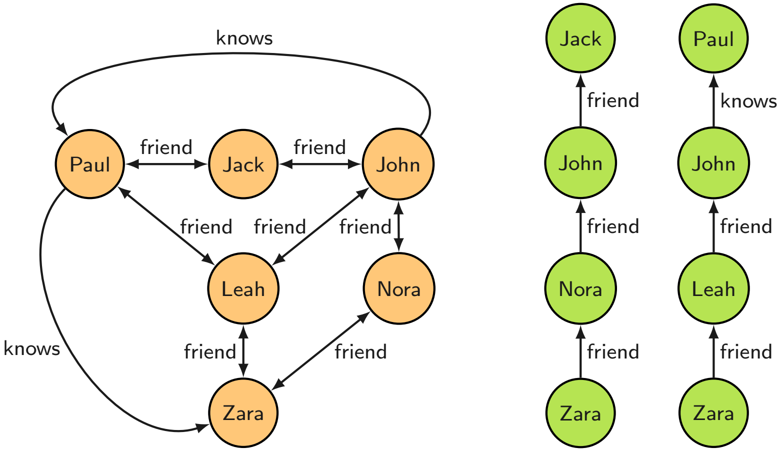

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- GEN: generate uniformly at random a path p in G such that p conforms to r and the length of p is n

Is COUNT a difficult problem?

Not surprisingly the answer is yes

We would like to give a precise characterization of the complexity of this problem

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  - Example: given as input a graph G, count the number of subgraphs G' of G such that G' is 3-colorable
  - $\#\mathrm{P} \subseteq \mathrm{SpanP}$
- SpanL : Count the number of distinct outputs of an NL-transducer
  - $\#$NFA: given as input an NFA A and a length n, count the number of words of length n accepted by A

#NFA is a hard problem [AJ93]

- SpanL is contained in #P
- SpanL is a hard class: if every function in SpanL can be computed in polynomial time, then P = NP
- #NFA is SpanL-complete under parsimonious reductions

COUNT is SpanL-complete

Parsimonious reduction from #NFA to COUNT

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COUNT and #NFA are equivalent problems

- In particular, in terms of the existence of efficient approximation algorithms

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- Best known approximation algorithm for #NFA worked in quasi-polynomial time [KSM95]

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- A fully polynomial-time almost-uniform generator [JVV86]

Questions?

The main ideas behind the solution

Our goal is to construct a fully polynomial-time randomized approximation scheme (FPRAS) for #NFA

The definition of #NFA:

Input	:	An NFA A over the alphabet $\{0, 1\}$ and a length n (given in unary)
Output	:	Number of words w such that $w \in \mathcal{L}(A)$ and $\lvert w \rvert = n$

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Assume that $\mathcal{L}_n(A) = \{w \in \mathcal{L}(A) \mid \lvert w \rvert = n\}$, so that the output of #NFA is $\lvert \mathcal{L}_n(A) \rvert$

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Moreover, number N has to be computed in time $\text{poly}(m, n, \frac{1}{\varepsilon})$, where m is the number of states of A

Constructing an FPRAS for #NFA

Assume that $A = (Q, \{0,1\}, \Delta, I, F)$

- Q is a finite set of states
- $\Delta \subseteq Q \times \{0,1\} \times Q$ is the transition relation
- $I \subseteq Q$ is a set of initial states
- $F \subseteq Q$ is a set of final states

First component: unroll automaton A

Construct A_{unroll} from A:

- for each state $q \in Q$, include copies q^0, q^1, ..., q^n in A_{unroll}
- for each transition $(p, a, q) \in \Delta$ and $i \in \{0, 1, \dots, n-1\}$, include transition (p^i, a, q^{i+1}) in A_{unroll}

Besides, eliminate from A_{unroll} unnecessary states: each state q^i is reachable from an initial state p^0 ($p \in I$)

Second component: a sketch to be used in the estimation

Define $\mathcal{L}(q^i)$ as the set of strings w such that there is a path from an initial state p^0 to q^i labeled with w

- Notice that $|w| = i$

Besides, define for every $X \subseteq Q$:

$$\mathcal{L}(X^i) = \bigcup_{q \in X} \mathcal{L}(q^i)$$

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Then the task is to compute an estimation of $|\mathcal{L}(F^n)|$

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Let $\kappa = \lceil \frac{nm}{\varepsilon} \rceil$, where $m = |Q|$

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We maintain for each state q^i:

- $N(q^i)$: a $(1 \pm \kappa^{-2})^i$-approximation of $|\mathcal{L}(q^i)|$
- $S(q^i)$: a multiset of uniform samples from $\mathcal{L}(q^i)$ of size $2\kappa^7$

Data structure to be inductively computed:

$$\text{sketch}[i] \quad = \quad \{N(q^j), S(q^j) \mid 0 \leq j \leq i \text{ and } q \in Q\}$$

The algorithm template

1. Construct A_{unroll} from A
2. For each state $q \in I$, set $N(q^0) = |\mathcal{L}(q^0)| = 1$ and $S(q^0) = \mathcal{L}(q^0) = \{\lambda\}$
3. For each $i = 0, \ldots, n-1$ and state $q \in Q$:
   (a) Compute $N(q^{i+1})$ given sketch[i]
   (b) Sample polynomially many uniform elements from $\mathcal{L}(q^{i+1})$ using $N(q^{i+1})$ and sketch[i], and let $S(q^{i+1})$ be the multiset of uniform samples obtained
4. Return an estimation of $|\mathcal{L}(F^n)|$ given sketch[n]

Computing an estimation $N(F^n)$ of $|\mathcal{L}(F^n)|$

We use notation $N(X^i)$ for an estimation $|\mathcal{L}(X^i)|$

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But the following holds, given a linear order $<$ on Q:

$$|\mathcal{L}(X^i)| \quad = \quad \sum_{p \in X} \Big|\mathcal{L}(p^i) \setminus \bigcup_{q \in X \,:\, q < p} \mathcal{L}(q^i)\Big|$$

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Computing an estimation $N(X^i)$ of $|\mathcal{L}(X^i)|$

$N(X^i)$ can be computed in polynomial time in the size of sketch[i]

- $S(p^i) \setminus \bigcup_{q \in X \,:\, q < p} \mathcal{L}(q^i)$ is constructed by checking for each $w \in S(p^i)$ whether w is not in $\mathcal{L}(q^i)$ for every $q \in X$ with $q < p$

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What guarantees that $N(X^i)$ is a good estimation of $|\mathcal{L}(X^i)|$?

The main property to maintain

$\mathcal{E}(i)$ holds if for every $p \in Q$ and $X \subseteq Q$:

$$\left| \frac{\left|\mathcal{L}(p^i) \setminus \bigcup_{q \in X} \mathcal{L}(q^i)\right|}{\left|\mathcal{L}(p^i)\right|} - \frac{\left|S(p^i) \setminus \bigcup_{q \in X} \mathcal{L}(q^i)\right|}{\left|S(p^i)\right|} \right| < \frac{1}{\kappa^3}$$

The use of the main property

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3. For each $i = 0, \ldots, n-1$ and state $q \in Q$:
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Proposition

If $\mathcal{E}(i)$ holds and $N(p^i)$ is a $(1 \pm \kappa^{-2})^i$-approximation of $|\mathcal{L}(p^i)|$ for every $p \in Q$, then $N(X^i)$ is a $(1 \pm \kappa^{-2})^{i+1}$-approximation of $|\mathcal{L}(X^i)|$ for every $X \subseteq Q$

The use of the main property

$\mathcal{E}(0)$ holds and $N(p^0)$ is a $(1 \pm \kappa^{-2})^0$-approximation of $|\mathcal{L}(p^0)|$ for every $p \in Q$

- Recall that $N(p^0) = |\mathcal{L}(p^0)|$ and $S(p^0) = \mathcal{L}(p^0)$ for every $p \in Q$

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- We want to use the values $N(X^0)$ to estimate the values $N(p^1)$

The use of the main property

For $p \in Q$, define:

$$
\begin{aligned}
Y &= \{q^0 \mid (q^0, 0, p^1) \text{ is a transition in } A_{unroll}\} \\
Z &= \{q^0 \mid (q^0, 1, p^1) \text{ is a transition in } A_{unroll}\}
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Then $\mathcal{L}(p^1) = \mathcal{L}(Y) \cdot \{0\} \uplus \mathcal{L}(Z) \cdot \{1\}$

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- So that $|\mathcal{L}(p^1)| = |\mathcal{L}(Y)| + |\mathcal{L}(Z)|$

The use of the main property

For $p \in Q$, define:

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Then $\mathcal{L}(p^1) = \mathcal{L}(Y) \cdot \{0\} \uplus \mathcal{L}(Z) \cdot \{1\}$

- So that $|\mathcal{L}(p^1)| = |\mathcal{L}(Y)| + |\mathcal{L}(Z)|$

Hence, given that $N(Y)$ is a $(1 \pm \kappa^{-2})$-approximation of $|\mathcal{L}(Y)|$ and $N(Z)$ is a $(1 \pm \kappa^{-2})$-approximation of $|\mathcal{L}(Z)|$:

$$N(Y) + N(Z) \text{ is a } (1 \pm \kappa^{-2})\text{-approximation of } N(p^1)$$

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$N(X^0)$ is a $(1 \pm \kappa^{-2})^1$-approximation of $|\mathcal{L}(X^0)|$ for every $X \subseteq Q$

$\Downarrow$

$N(p^1) = N(R_0(p^1)) + N(R_1(p^1))$ is a $(1 \pm \kappa^{-2})^1$-approximation of $N(p^1)$ for every $p \in Q$

where $R_b(p^1) = \{q^0 \mid (q^0, b, p^1) \text{ is a transition in } A_{unroll}\}$

The use of the main property: a summary

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$\mathcal{E}(1)$ holds and $N(p^1)$ is a $(1 \pm \kappa^{-2})^1$-approximation of $|\mathcal{L}(p^1)|$ for every $p \in Q$

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$\mathcal{E}(1)$ holds and $N(p^1)$ is a $(1 \pm \kappa^{-2})^1$-approximation of $|\mathcal{L}(p^1)|$ for every $p \in Q$

$\Downarrow$

$N(X^1)$ is a $(1 \pm \kappa^{-2})^2$-approximation of $|\mathcal{L}(X^1)|$ for every $X \subseteq Q$

$\Downarrow$

$N(p^2) = N(R_0(p^2)) + N(R_1(p^2))$ is a $(1 \pm \kappa^{-2})^2$-approximation of $N(p^2)$ for every $p \in Q$

where $R_b(p^2) = \{q^1 \mid (q^1, b, p^2) \text{ is a transition in } A_{unroll}\}$

The final result

Proposition

If $\mathcal{E}(i)$ *holds for every* $i \in \{0, 1, \ldots, n\}$, *then* $N(F^n)$ *is a* $(1 \pm \varepsilon)$*-approximation of* $|\mathcal{L}(F^n)|$

Questions?

How can we maintain property $\mathcal{E}(i)$?

Sampling from a state

We need to construct the multiset $S(q^{i+1})$ of uniform samples

Recall that:

- $S(q^{i+1})$ contains $2\kappa^7$ words from $\mathcal{L}(q^{i+1})$
- $S(q^{i+1})$ is computed assuming that $N(q^{i+1})$ and $\text{sketch}[i] = \{N(q^j), S(q^j) \mid 0 \leq j \leq i\}$ have already been constructed

To recall

1. Construct A_{unroll} from A

2. For each state $q \in I$, set $N(q^0) = |\mathcal{L}(q^0)| = 1$ and $S(q^0) = \mathcal{L}(q^0) = \{\lambda\}$

3. For each $i = 0, \ldots, n-1$ and state $q \in Q$:

   (a) Compute $N(q^{i+1})$ given sketch[i]

   (b) Sample polynomially many uniform elements from $\mathcal{L}(q^{i+1})$ using $N(q^{i+1})$ and sketch[i], and let $S(q^{i+1})$ be the multiset of uniform samples obtained

4. Return an estimation of $|\mathcal{L}(F^n)|$ given sketch[n]

Sampling from q^{i+1}

To generate a sample in $\mathcal{L}(q^{i+1})$, we construct a sequence w^{i+1}, w^i, ..., w^1, w^0 such that

- $w^{i+1} = \lambda$
- $w^j = b_j w^{j+1}$ with $b_j \in \{0, 1\}$
- $w^0 \in \mathcal{L}(q^{i+1})$

Sampling from q^{i+1}

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- $w^0 \in \mathcal{L}(q^{i+1})$

To choose $w^i = bw^{i+1}$, construct for $b = 0, 1$:

$$P_b \quad = \quad \{p^i \mid (p^i, b, q^{i+1}) \text{ is a transition in } A_{unroll}\}$$

Sampling from q^i

P_0 and P_1 are sets of states at layer i

We can compute $N(P_0)$ and $N(P_1)$ as follows:

$$N(X^i) = \sum_{p \in X} N(p^i) \frac{\left| S(p^i) \setminus \bigcup_{q \in X \,:\, q < p} \mathcal{L}(q^i) \right|}{\left| S(p^i) \right|}$$

Sampling from q^i

P_0 and P_1 are sets of states at layer i

We can compute $N(P_0)$ and $N(P_1)$ as follows:

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We choose $b \in \{0, 1\}$ with probability:

$$\frac{N(P_b)}{N(P_0) + N(P_1)}$$

We could have started from a set of states

The previous procedure works for every set of states P^{i+1}:

$$P_b \quad = \quad \{p^i \mid \exists r^{i+1} \in P^{i+1} : (p^i, b, r^{i+1}) \text{ is a transition in } A_{unroll}\}$$

In particular, we applied the procedure for $P^{i+1} = \{q^{i+1}\}$

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In particular, we applied the procedure for $P^{i+1} = \{q^{i+1}\}$

The following recursive procedure summarizes the previous idea:

$$\textbf{Sample}(i+1, \{q^{i+1}\}, \lambda, \varphi_0)$$

It uses sets of states $P^{i+1} = \{q^{i+1}\}$, P^i, ..., P^1, P^0 and an initial probability φ_0

The sampling algorithm

Sample(j, P^j, w^j, φ)

1. If $j = 0$, then with probability φ return w^0, otherwise return **fail**
2. Compute $P_{j,b} = \{p^{j-1} \mid \exists r^j \in P^j : (p^{j-1}, b, r^j)$ is a transition in $A_{unroll}\}$ for $b = 0, 1$
3. Choose $b \in \{0, 1\}$ with probability $p_b = \dfrac{N(P_{j,b})}{N(P_{j,0}) + N(P_{j,1})}$
4. Set $P^{j-1} = P_{j,b}$ and $w^{j-1} = bw^j$
5. Return **Sample**$(j-1, P^{j-1}, w^{j-1}, \frac{\varphi}{p_b})$

The key observation

Let $x = x_1 \cdots x_{i+1}$ be a word in $\mathcal{L}(q^{i+1})$

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Let $x = x_1 \cdots x_{i+1}$ be a word in $\mathcal{L}(q^{i+1})$

We have that:

$$\begin{aligned}
&\mathbf{Pr}(\text{the output of } \textbf{Sample} \text{ is } x) \\
&\quad = \mathbf{Pr}(w^0 = x \wedge \text{ the last call to } \textbf{Sample} \text{ does not fail}) \\
&\quad = \mathbf{Pr}(\text{the last call to } \textbf{Sample} \text{ does not fail} \mid w^0 = x) \cdot \mathbf{Pr}(w^0 = x) \\
&\quad = \left(\left(\prod_{j=1}^{i+1} \frac{N(P_{j,x_j})}{N(P_{j,0}) + N(P_{j,1})} \right)^{-1} \cdot \varphi_0 \right) \cdot \left(\prod_{j=1}^{i+1} \frac{N(P_{j,x_j})}{N(P_{j,0}) + N(P_{j,1})} \right) \\
&\quad = \varphi_0
\end{aligned}$$

The value of the initial probability φ_0

Proposition

Assume that $\mathcal{E}(j)$ holds for each $j < i+1$. If w is the output of $\textbf{Sample}(i+1, \{q^{i+1}\}, \lambda, \frac{e^{-5}}{N(q^{i+1})})$, *then*

- $\varphi \in (0,1)$ *in every recursive call to* **Sample**
- $\textbf{Pr}(w = \textbf{fail}) \leq 1 - e^{-9}$
- $\textbf{Pr}(w = x) = \dfrac{e^{-5}}{N(q^{i+1})}$ *for every* $x \in \mathcal{L}(q^{i+1})$

The value of the initial probability φ_0

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Assume that $\mathcal{E}(j)$ holds for each $j < i+1$. If w is the output of **Sample**$(i+1, \{q^{i+1}\}, \lambda, \frac{e^{-5}}{N(q^{i+1})})$, *then*

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- $\mathbf{Pr}(w = x) = \dfrac{e^{-5}}{N(q^{i+1})}$ *for every* $x \in \mathcal{L}(q^{i+1})$

Hence, conditioned on not failing, **Sample**$(i+1, \{q^{i+1}\}, \lambda, \frac{e^{-5}}{N(q^{i+1})})$ returns a uniform sample from $\mathcal{L}(q^{i+1})$

Bounding the probability of breaking the main assumption

Recall that $\mathcal{E}(i)$ holds if for every $q \in Q$ and $X \subseteq Q$:

$$\left| \frac{\left|\mathcal{L}(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)\right|}{\left|\mathcal{L}(q^i)\right|} - \frac{\left|S(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)\right|}{\left|S(q^i)\right|} \right| < \frac{1}{\kappa^3}$$

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$$\left| \frac{\left|\mathcal{L}(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)\right|}{\left|\mathcal{L}(q^i)\right|} - \frac{\left|S(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)\right|}{\left|S(q^i)\right|} \right| < \frac{1}{\kappa^3}$$

By using Hoeffding's inequality, it is possible to conclude that:

$$\mathbf{Pr}\left(\bigwedge_{j=0}^{n} \mathcal{E}(j) \right) \geq 1 - e^{-\kappa}$$

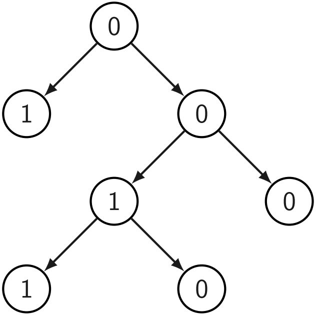
The complete algorithm: final comments

Putting all together, we obtain that the probability that the algorithm returns a wrong estimate is at most $\frac{1}{4}$

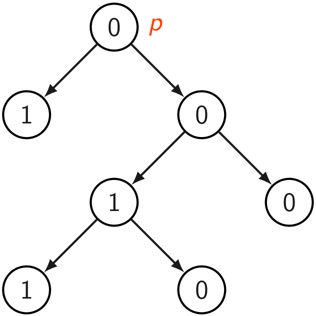
The algorithm runs in time poly$(m, n, \frac{1}{\varepsilon})$

The extension of the approach to tree automata

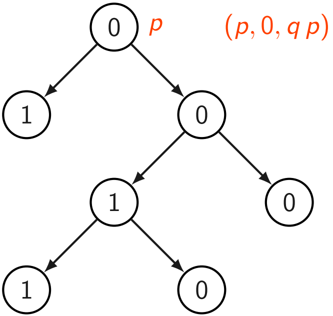
Tree automata



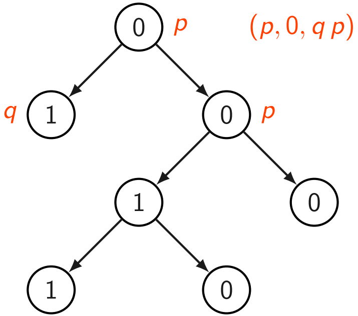
Tree automata



Tree automata



Tree automata



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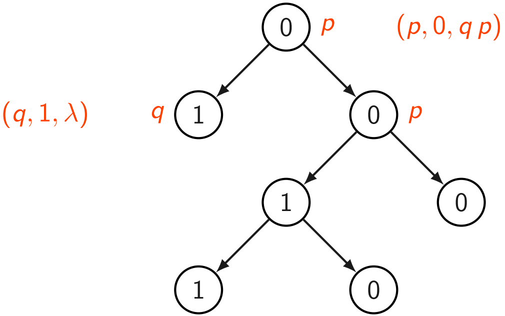

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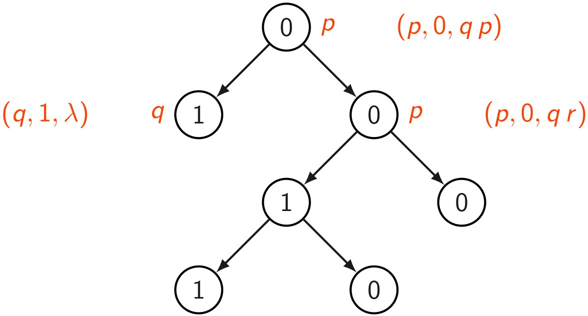
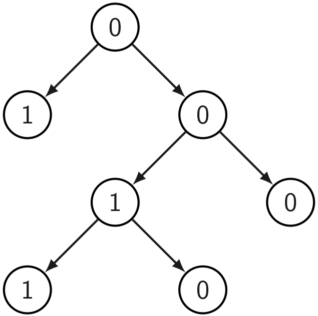


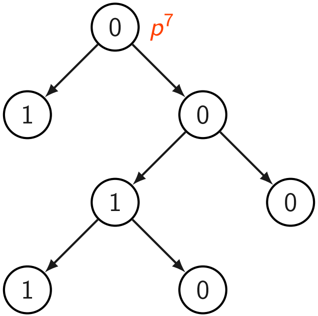
The problem $\#$TA

Input	:	A tree automaton (TA) T over the alphabet $\{0, 1\}$ and a number n (given in unary)
Output	:	Number of trees t such that $t \in \mathcal{L}(T)$ and the number of nodes of t is n

Constructing an FPRAS for #TA



Constructing an FPRAS for #TA



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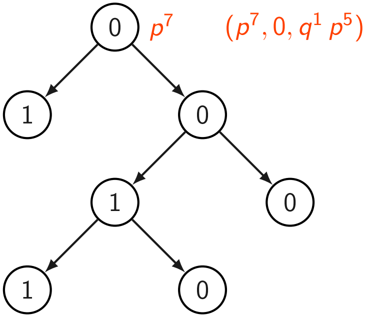

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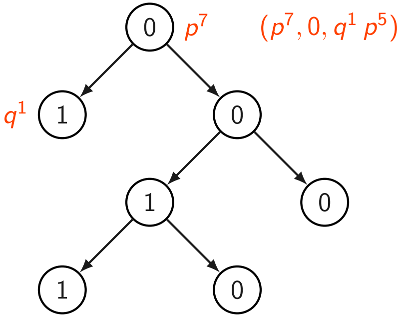

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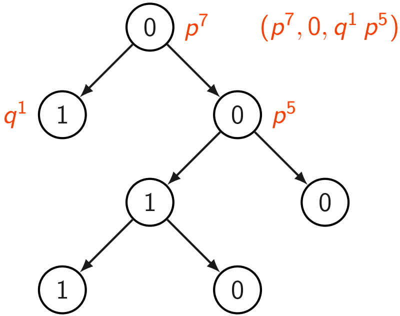

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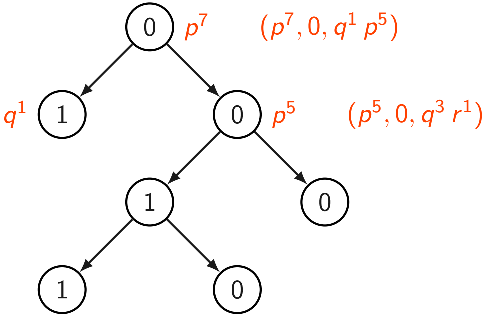

Reducing to a counting problem for *succinct* NFA

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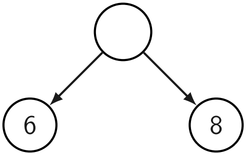

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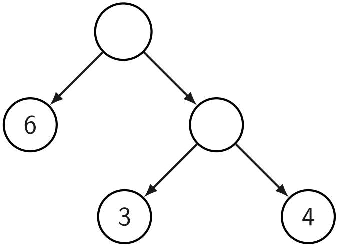

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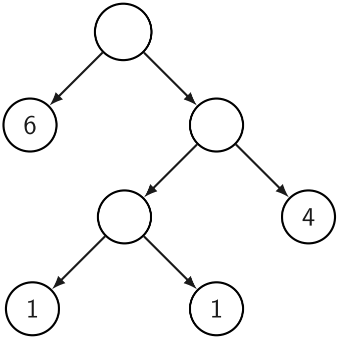

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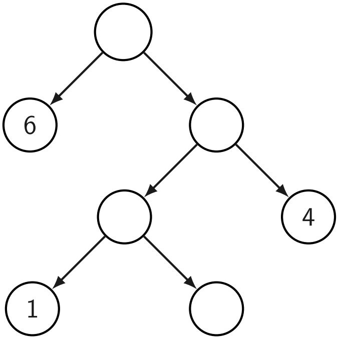

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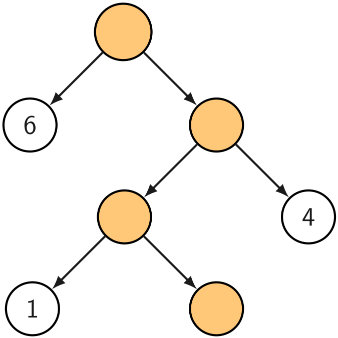

Succinct NFA $S = (Q, \Sigma, \Delta, I, F)$

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- The length of $w_1 \cdots w_n$ is n

The main result about #Succinct-NFA

The definition of #Succinct-NFA:

Input	:	A succinct NFA S and a length n (given in unary)
Output	:	Number of words w such that $w \in \mathcal{L}(S)$ and $\lvert w \rvert = n$

The main result about #Succinct-NFA

The definition of #Succinct-NFA:

Input	:	A succinct NFA S and a length n (given in unary)
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Theorem

#Succinct-NFA admits an FPRAS when restricted to the class of succinct NFA $S = (Q, \Sigma, \Delta, I, F)$ such that for every $(p, A, q) \in \Delta$, there exists an oracle which can:

(1) *test membership in A,*

(2) *produce an estimate of the size of $|A|$, and*

(3) *generate almost-uniform samples from A.*

The main result about #TA

Theorem

#TA admits an FPRAS

The main corollaries about query answering

Let #ACQ:

Input	:	A database D and an acyclic conjunctive query Q
Output	:	$\lvert Q(D)\rvert$

The main corollaries about query answering

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Input	:	A database D and an acyclic conjunctive query Q
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Corollary

#ACQ admits an FPRAS

The main corollaries about query answering

Given $k \geq 1$, let $\#k$-HW:

Input	:	A database D and a conjunctive query Q such that the hypertree width of Q is at most k
Output	:	$\lvert Q(D) \rvert$

The main corollaries about query answering

Given $k \geq 1$, let $\#k$-HW:

Input	:	A database D and a conjunctive query Q such that the hypertree width of Q is at most k
Output	:	$\lvert Q(D) \rvert$

Corollary

$\#k$-HW admits an FPRAS

Some final remarks

The results presented in this talk are proved in [ACJR21a] and [ACJR21b]

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Future work:

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- Make algorithms practical

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Future work:

- Make algorithms practical
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- Can the results be extended for semantic acyclicity?

The results presented in this talk are proved in [ACJR21a] and [ACJR21b]

Future work:

- Make algorithms practical
  - Implementation of FPRAS for #NFA works well with smaller bounds
- Can the results be extended for semantic acyclicity?
- Can the results be extended to context free grammars?

The results presented in this talk are proved in [ACJR21a] and [ACJR21b]

Future work:

- Make algorithms practical
    - Implementation of FPRAS for #NFA works well with smaller bounds
- Can the results be extended for semantic acyclicity?
- Can the results be extended to context free grammars?
    - Only quasi-polynomial time approximation are known for this problem [GJKSM97]

Questions?

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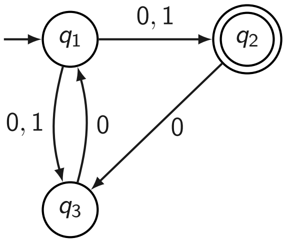
Appendix

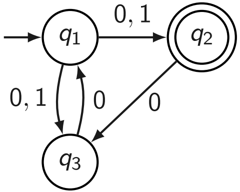
COUNT is SpanL-complete (under parsimonious reductions)

Consider the following NFA A:

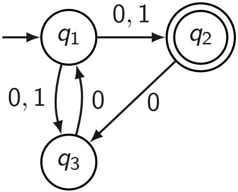


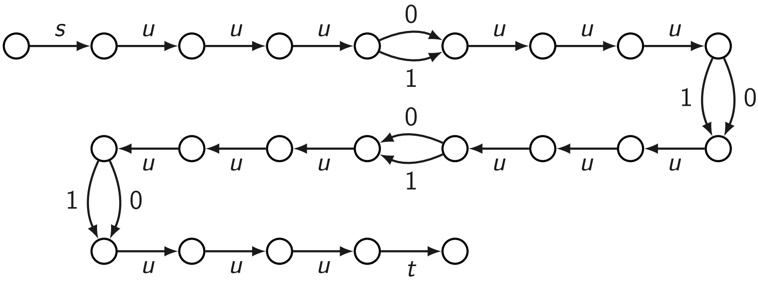
Assume we need to return the number of words of length 4 accepted by A

Reduction from #NFA to COUNT

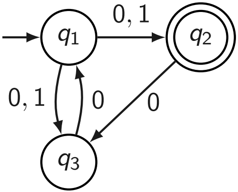


Reduction from #NFA to COUNT

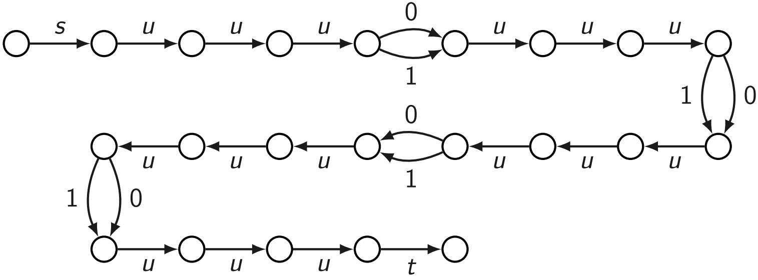




Reduction from #NFA to COUNT



q_i is an initial state : s/u^i



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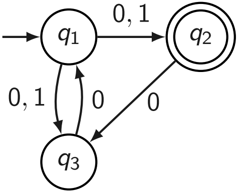

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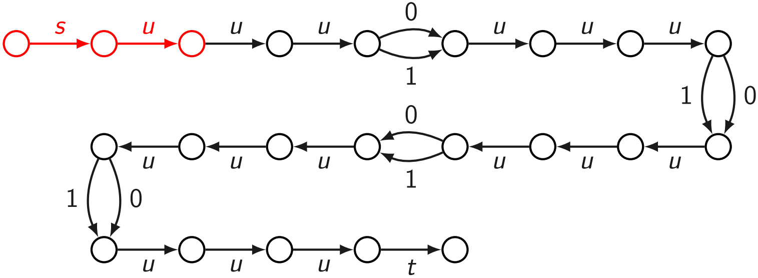
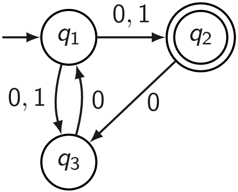


Reduction from #NFA to COUNT



q_i is an initial state	:	s/u^i
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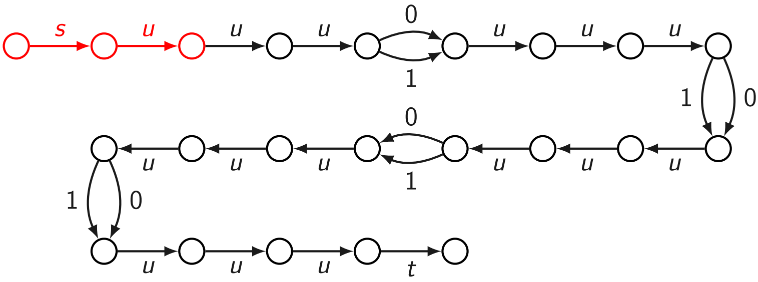

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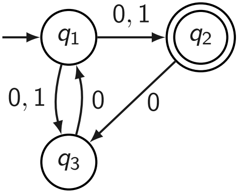

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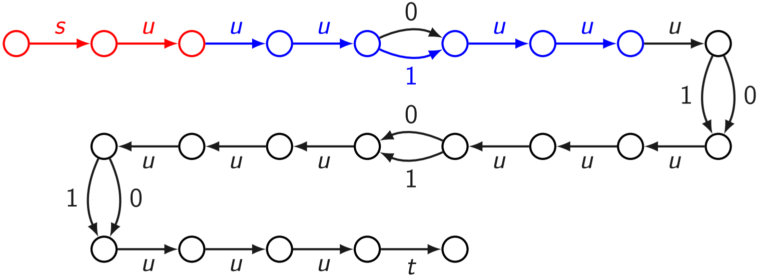

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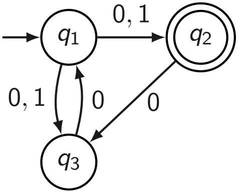

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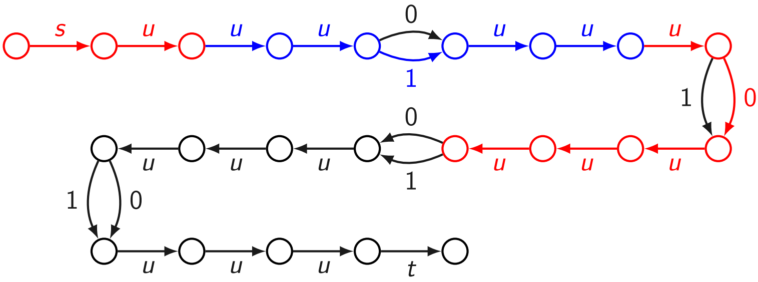

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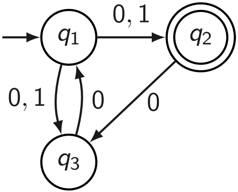

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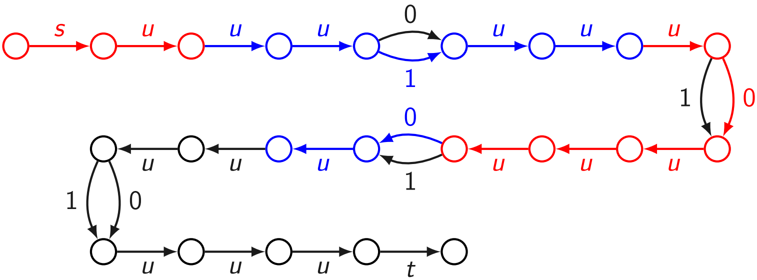

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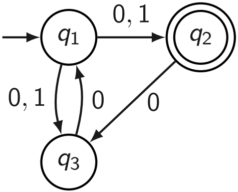

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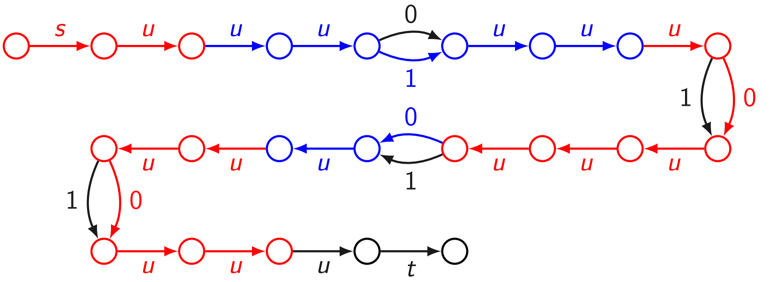

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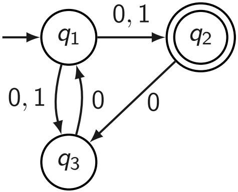

$$
\begin{array}{rcl}
q_i \text{ is an initial state} & : & s/u^i \\
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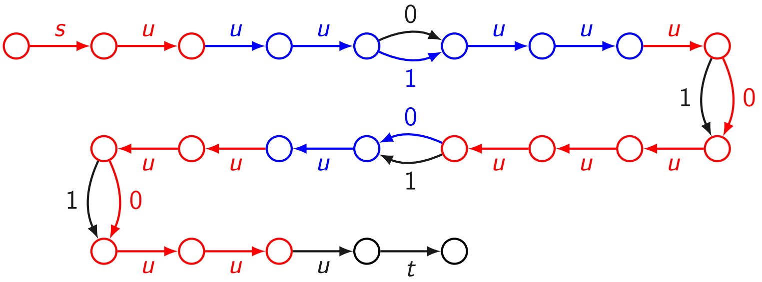
q_i \text{ is a final state} & : & u^{3-i}/t
\end{array}
$$



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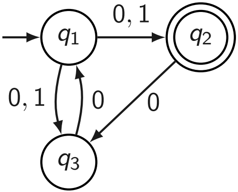

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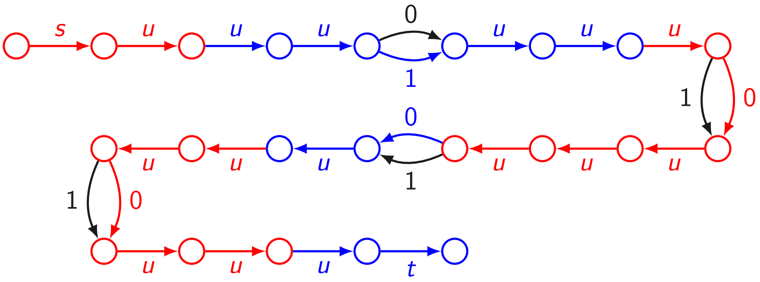

Reduction from #NFA to COUNT

Define $r = (s/u + u/u/1/u/u/ + u/0/u/u/u + 0/u + \ldots + u/t)^*$

Reduction from #NFA to COUNT

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Number of words of length 4 accepted by A

=

Number of paths p in G such that p conforms to r and the length of p is $21 = (5 \times 3 + 4 + 2)$

Bounding the probability of breaking the main assumption

Recall that $\mathcal{E}(i)$ holds if for every $q \in Q$ and $X \subseteq Q$:

$$\left| \frac{\left|\mathcal{L}(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)\right|}{\left|\mathcal{L}(q^i)\right|} - \frac{\left|S(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)\right|}{\left|S(q^i)\right|} \right| < \frac{1}{\kappa^3}$$

We know that $\mathcal{E}(0)$ holds.

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We know that $\mathcal{E}(0)$ holds. We need to compute a lower bound for:

$$\mathbf{Pr}\left(\bigwedge_{j=0}^{n} \mathcal{E}(j) \right)$$

Bounding the probability of breaking $\mathcal{E}(i)$

Assume that $\bigwedge_{j=0}^{i-1} \mathcal{E}(j)$ holds

Let $q \in Q$ and $S(q^i)$ be a multiset of $2\kappa^7$ samples from $\mathcal{L}(q^i)$ computed by calling **Sample**$(i, \{q^i\}, \lambda, \frac{e^{-5}}{N(q^i)})$

- Each element of $S(q^i)$ is obtained by repeatedly calling **Sample** until the output is different from **fail**

Assume that $S(q^i) = \{w_1, \ldots, w_t\}$ with $t = 2\kappa^7$

Bounding the probability of breaking $\mathcal{E}(i)$

Let $X \subseteq Q$, and Y_i be a Bernoulli random variable for $i \in \{1, \ldots, t\}$:

$$Y_i = 1 \quad \text{if and only if} \quad w_i \in \left(\mathcal{L}(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i) \right)$$

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We have that:

$$\begin{aligned}
\mathbb{E}[Y_i] &= \frac{|\mathcal{L}(q^i) \smallsetminus \bigcup_{p \in X} \mathcal{L}(p^i)|}{|\mathcal{L}(q^i)|} \\
\sum_{j=1}^{t} Y_i &= |S(q^i) \smallsetminus \bigcup_{p \in X} \mathcal{L}(p^i)| \\
t &= |S(q^i)|
\end{aligned}$$

By using Hoeffding's inequality

$$\mathbf{Pr}\Bigg(\Bigg|\frac{|S(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)|}{|S(q^i)|} - \frac{|\mathcal{L}(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)|}{|\mathcal{L}(q^i)|}\Bigg| \geq \frac{1}{\kappa^3} \ \Bigg| \bigwedge_{j=0}^{i-1} \mathcal{E}(j)\Bigg)$$

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$$\mathbf{Pr}\left(\left|\frac{1}{t}\sum_{j=1}^{t} Y_i - \mathbb{E}\left[\frac{1}{t}\sum_{j=1}^{t} Y_i\right]\right| \geq \frac{1}{\kappa^3} \;\middle|\; \bigwedge_{j=0}^{i-1} \mathcal{E}(j)\right)$$

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$$= 2e^{-2\left(\frac{1}{\kappa^6}\right)2\kappa^7}$$

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$$= 2e^{-4\kappa}$$

By taking the union bound

$$\mathbf{Pr}\Bigg(\exists q \in Q\, \exists X \subseteq Q \left| \frac{|S(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)|}{|S(q^i)|} - \frac{|\mathcal{L}(q^i) \setminus \bigcup_{p \in X} \mathcal{L}(p^i)|}{|\mathcal{L}(q^i)|} \right| \geq \frac{1}{\kappa^3} \;\Bigg|\; \bigwedge_{j=0}^{i-1} \mathcal{E}(j) \Bigg) \leq$$

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$$m2^m 2e^{-4\kappa} \leq \kappa 2^\kappa 2e^{-4\kappa} \leq 2e^{-2\kappa}$$

The conclusion

Rewriting the previous result:

$$\mathbf{Pr}\left(\mathcal{E}(i) \;\middle|\; \bigwedge_{j=0}^{i-1} \mathcal{E}(j)\right) \;\geq\; 1 - e^{-2\kappa}$$

We conclude that:

$$\mathbf{Pr}\left(\bigwedge_{j=0}^{n} \mathcal{E}(j)\right) \;\geq\; 1 - e^{-\kappa}$$

The complete algorithm

Input: NFA $A = (Q, \{0,1\}, \Delta, I, F)$ with $m = |Q|$, length n given in unary and error $\varepsilon \in (0,1)$

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4. For each $q^0 \in I^0$, set $N(q^0) = 1$ and $S(q^0) = \{\lambda\}$

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6. Return $N(F^n)$ as an estimation of $|\mathcal{L}_n(A)|$

The complete algorithm: final comments

The probability that the algorithm returns a wrong estimate is at most $\frac{1}{4}$

- Considering $c(\kappa) = \lceil \frac{2+\log(4)+8\log(\kappa)}{\log(1-e^{-9})^{-1}} \rceil$

The algorithm runs in time poly$(m, n, \frac{1}{\varepsilon})$