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Annals of Pure and Applied Logic 152 (2008) 3–30

ANNALS OF PURE AND APPLIED LOGIC

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Game-based notions of locality over finite models☆

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Abstract

Locality notions in logic say that the truth value of a formula can be determined locally, by looking at the isomorphism type of a small neighbourhood of its free variables. Such notions have proved to be useful in many applications. They all, however, refer to isomorphisms of neighbourhoods, which most local logics cannot test. A stronger notion of locality says that the truth value of a formula is determined by what the logic itself can say about that small neighbourhood. Since the expressiveness of many logics can be characterized by games, one can also say that the truth value of a formula is determined by the type, with respect to a game, of that small neighbourhood. Such game-based notions of locality can often be applied when traditional isomorphism-based notions of locality cannot.

Our goal is to study game-based notions of locality. We work with an abstract view of games that subsumes games for many logics. We look at three, progressively more complicated locality notions. The easiest requires only very mild conditions on the game and works for most logics of interest. The other notions, based on Hanf's and Gaifman's theorems, require more restrictions. We state those restrictions and give examples of logics that satisfy and fail the respective game-based notions of locality.

MSC: 03C07; 03C13; 03C80; 03C98

Keywords: Finite model theory; Gaifman locality; Hanf locality; Games

1. Introduction

Locality is a property of logics that finds its origins in the work by Hanf [10] and Gaifman [8], and that was shown to be very useful in the context of finite model theory. Locality is primarily used in two ways: for proving inexpressibility results over finite structures, where most traditional model-theoretic tools fail, and for establishing normal forms for logical formulae. The former has led to new easy winning strategies in logical games [4,6,18], with applications in descriptive complexity (e.g., the study of monadic NP and its relatives [6], or circuit complexity classes

☆ This is the full version of two extended abstracts presented at the *Computer Science Logic* Conference in 2004, and *Workshop on Logic, Language, Information, and Computation* in 2005.

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doi:10.1016/j.apal.2007.11.012

[19]), in databases (e.g., establishing bounds on the expressiveness of aggregate queries [13], or on query rewriting in data integration and exchange [5,1]), and in formal languages (e.g., in characterizing subclasses of star-free languages [25]). Local normal forms like those in [8,23] have found many applications as well, for example, in the design of low-complexity model-checking algorithms [7,9,24], in automata theory [23] and in computing weakest preconditions for database transactions [2].

There are two closely related ways of stating the locality of logical formulae. One, originating in Hanf's work [10], says that if two structures $\mathfrak{A}$ and $\mathfrak{B}$ realize the same multiset of types of neighbourhoods of radius d, then they agree on a given sentence Φ. Here d depends only on Φ, and not on the structure. The other, inspired by Gaifman's theorem [8], says that if the d-neighbourhoods of two tuples $\bar{a}_1$ and $\bar{a}_2$ in a structure $\mathfrak{A}$ are isomorphic, then $\mathfrak{A} \models \varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$. Again, d depends on φ, and not on $\mathfrak{A}$.

If all formulae in a logic are local, it is easy to prove bounds on its expressive power. For example, connectivity violates the Hanf notion of locality, as one cycle of length $2m$ and two disjoint cycles of length m realize the same multiset of types of neighbourhoods of radius d as long as $m > 2d + 1$. Likewise, the transitive closure of a graph violates the Gaifman notion of locality. For instance, in the directed graph with edges (a_i, a_{i+1}) for $0 \leq i \leq n-1$, radius-d neighbourhoods of (a_d, a_{n-d}) and (a_{n-d}, a_d) are isomorphic as long as $n > 4d + 1$, and yet these pairs are distinguished by the transitive closure query.

These notions of locality, while very useful in many applications, have one deficiency: they all refer to the *isomorphism* of neighbourhoods, which is a very strong property (typically not expressible in a logic that satisfies one of the locality properties). There are situations when these notions are not applicable simply because structures do not have enough isomorphic neighbourhoods. One example was given in [19], which discussed applicability of locality techniques to the study of small parallel complexity classes: consider a directed tree in which all non-leaf nodes have different out-degrees. Then locality techniques cannot be used to derive any results about logics over such trees.

Intuitively, though, it seems that requiring the isomorphism of neighbourhoods is too much. Suppose we are dealing with first-order logic FO, which is local in the sense of Gaifman. For a structure $\mathfrak{A}$, it appears that if FO itself cannot see the difference between two large enough neighbourhoods of points a and b in $\mathfrak{A}$, then it should not be able to see the difference between elements a and b in $\mathfrak{A}$. That is, for a given formula $\varphi(x)$, if radius-d neighbourhoods of a and b cannot be distinguished by sufficiently many FO formulae, then $\mathfrak{A} \models \varphi(a) \leftrightarrow \varphi(b)$. Gaifman's theorem [8] actually implies that this is the case: if φ is of quantifier rank k, then there exist numbers d and ℓ, dependent on k only, such that if radius-d neighbourhoods of a and b cannot be distinguished by formulae of quantifier rank ℓ, then $\mathfrak{A} \models \varphi(a) \leftrightarrow \varphi(b)$.

In general, it seems that if a logic is local (say, in the sense of Gaifman), then for each formula φ there is a number d such that if the logic cannot distinguish radius-d neighbourhoods of $\bar{a}$ and $\bar{b}$, then $\varphi(\bar{a}) \leftrightarrow \varphi(\bar{b})$.

The goal of this paper is to introduce such notions of locality based on the logical indistinguishability of neighbourhoods, and see if they apply to logics that are known to possess isomorphism-based locality properties. Since logical equivalence is often captured by Ehrenfeucht–Fraïssé-type of games, we shall refer to such new notions of locality as *game-based*. We shall discover that the situation is more complex than one may have expected, and passing from an isomorphism-based locality to game-based one is by no means guaranteed for logics known to possess the former.

These new game-based notions of locality can be applied when the traditional isomorphism-based notions cannot (which, in particular, makes it possible to show more bounds on the expressiveness of logics). This is demonstrated by the following example.

Example 1.1. Let σ be the vocabulary of a unary relation symbol U and a binary relation symbol E. Given a finite structure $\mathfrak{A}$, let $Q(\mathfrak{A})$ be the set of elements a in the universe of $\mathfrak{A}$ such that a is in U, and the number of elements c such that $(a, c) \in E$, is even.

It can be shown (e.g., by a direct game argument) that Q is not FO-definable. However, we cannot use any of the classical locality notions to prove this: even if neighbourhoods of a and b of radius 1 are isomorphic, it just means that a and b cannot be distinguished by Q.

Nonetheless, with game-based locality, the proof of inexpressibility of Q can be obtained easily. Such a notion (and we shall see later in the paper that it applies to FO) states that if Q were FO-definable, then there would be constants $r, \ell \geq 0$ such that for every structure $\mathfrak{A}$ and elements $a, b \in A$, if radius-r neighbourhoods of a and b cannot be

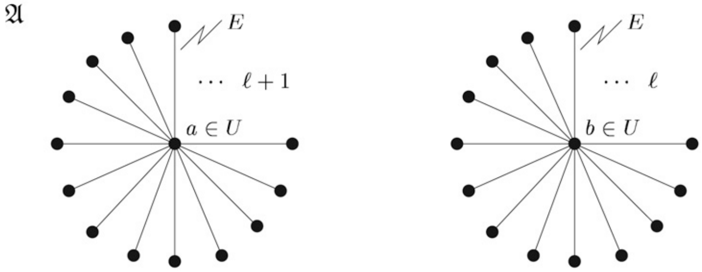

Fig. 1. Structure $\mathfrak{A}$ of Example 1.1.

distinguished by an ℓ-round Ehrenfeucht-Fraïssé game, then a and b cannot be distinguished by Q. But now look at $\mathfrak{A}$ in Fig. 1, where the E-relation is shown, and U is interpreted as $\{a, b\}$, and the number of c's connected to a and b is ℓ and $\ell + 1$, respectively. It is immediate then that r-neighbourhoods of a and b (which are radius-1 neighbourhoods) cannot be distinguished by an ℓ-round Ehrenfeucht-Fraïssé game, and yet a and b are distinguished by Q.

Organization of the paper. First, we present a unifying framework for talking about logical games. Our framework subsumes games for FO, and many of its counting and generalized quantifier extensions. We then see what conditions on games need to be imposed in order to recover game-based notions of locality. We look at three progressively more complex notions: weak locality, Gaifman-locality, and Hanf-locality, and state conditions on games under which they can be guaranteed. While weak locality requires very little, even that notion can fail in some unary-quantifier extensions of FO. Hanf-locality under games fails even for FO, but holds for a number of counting logics. Gaifman-locality under games holds for many logics, although the proofs are harder for weaker forms of counting.

2. Notation

We work with finite structures, whose universes are subsets of some countable infinite set U. All vocabularies will be finite sequences of relation symbols $\sigma = \langle R_1, \ldots, R_n\rangle$; a σ*-structure* $\mathfrak{A}$ consists of a finite universe $A \subset U$ and an interpretation of each m-ary relation symbol R_i in σ as a subset of A^m. We adopt the convention that the universe of a structure is denoted by the corresponding Roman letter, that is, the universe of $\mathfrak{A}$ is A, the universe of $\mathfrak{B}$ is B, etc. Isomorphism of structures will be denoted by $\cong$.

An *m-ary query* Q on σ-structures, $m \geq 0$, is a map, closed under isomorphism, that associates with each σ-structure $\mathfrak{A}$ a subset of A^m. We assume that 0-ary queries are maps from σ-structures to the Boolean values *true* and *false*. A logical formula $\varphi(x_1, \ldots, x_m)$ defines an m-ary query $Q_\varphi(\mathfrak{A}) = \{(a_1, \ldots, a_m) \mid \mathfrak{A} \models \varphi(a_1, \ldots, a_m)\}$.

We shall denote first-order logic by FO. For each $S \subseteq \mathbb{N}$, we let $\mathbf{Q}_S$ denote a *simple unary generalized quantifier* [17,26] that gives rise to the extension FO($\mathbf{Q}_S$) of FO with the following formation rule: if $\psi(x, \bar{y})$ is a formula, then $\varphi(\bar{y}) = \mathbf{Q}_S x\ \psi(x, \bar{y})$ is a formula. The semantics is as follows: $\mathfrak{A} \models \varphi(\bar{a})$ if $|\{b \mid \mathfrak{A} \models \psi(b, \bar{a})| \in S$. One could also define FO($\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_\ell}$) as FO extended with a collection $\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_\ell}$ of simple unary generalized quantifiers.

A well-known special case is that of *modulo quantifiers* $\mathbf{Q}_{\{np \mid n \in \mathbb{N}\}}$ (cf. [21,22,26]), which we shall denote by $\mathbf{Q}_p$. We shall also consider $\mathbf{Q}_{\{p \mid p \text{ is prime}\}}$, denoted by $\mathbf{Q}_{\text{PRIME}}$.

Finally, we define a powerful counting logic that subsumes most counting extensions of FO, in particular FO extended with arbitrary collections of unary generalized quantifiers. The structures for this logic are two-sorted, and the second sort is $\mathbb{N}$. There is a constant symbol for each $k \in \mathbb{N}$. The logic has *infinitary* connectives $\bigvee$ and $\bigwedge$, and *counting terms*: if φ is a formula and $\bar{x}$ a tuple of free first-sort variables in φ, then $\#\bar{x}.\varphi$ is a term of the second sort, whose free variables are those in φ except $\bar{x}$. Its value is the number of tuples $\bar{a}$ that make $\varphi(\bar{a}, \cdot)$ true. This logic, denoted by $\mathcal{L}_{\infty\omega}(\mathbf{Cnt})$, is too powerful as it defines all properties of finite structures, but we restrict it using the notion of quantifier rank $\mathsf{qr}(\cdot)$ (which is the maximum depth of quantifier nesting *excluding* quantification over the numerical universe for two-sorted logics). For $\mathcal{L}_{\infty\omega}(\mathbf{Cnt})$, we also define $\mathsf{qr}(\#\bar{x}.\varphi)$ as $\mathsf{qr}(\varphi) + |\bar{x}|$.

We now let $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ be $\mathcal{L}_{\infty\omega}(\mathbf{Cnt})$ restricted to formulae and terms that have finite quantifier rank. This logic subsumes known counting extensions of FO, but cannot express many properties definable, say, in fixed-point logics or fragments of second-order logic [18]. Notice also that $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ can express all numerical properties by means of infinitary disjunctions.

3. Games and logics

We shall be dealing with the locality of logics where indistinguishability of neighbourhoods is described in terms of winning strategies of games. Thus, our first goal is to present an abstract view of games that characterize the expressiveness of logics which are known to be local under the standard isomorphism-based notion.

All such games are played by two players, the *spoiler* and the *duplicator*, on two σ-structures $\mathfrak{A}$ and $\mathfrak{B}$. The goal of the spoiler is to show that the structures are different while the duplicator is trying to show that they are the same. More precisely, in each round i the players play two elements of the structures, $a_i \in A$ and $b_i \in B$. The duplicator wins after k rounds if the function $f : \{a_1, \dots, a_k\} \to \{b_1, \dots, b_k\}$ given by $f(a_i) = b_i$ is a partial isomorphism.

As our first example, we consider the *bijective game* of [11]. This game (which captures expressiveness of sentences of $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$) is played as follows. In round i, the duplicator selects a bijection $f_i : A \to B$ (if $|A| \neq |B|$, the duplicator loses the game). The spoiler then picks $a_i \in A$, and the duplicator is forced to respond with $f_i(a_i) \in B$.

We now look at the standard k-round Ehrenfeucht-Fraïssé game on $\mathfrak{A}$ and $\mathfrak{B}$. Recall that in this game in each round $i \leq k$, the spoiler chooses one of the two structures, let us say $\mathfrak{A}$, and an element in that structure, $a_i \in A$. The duplicator then responds with an element $b_i \in B$. We now notice that this standard description can be presented in a way that resembles the definition of bijective games. Namely, if the duplicator has a winning strategy in the k-round Ehrenfeucht-Fraïssé game, it means that in each round i, depending on the current position of the game, he has a way to respond to *any* move by the spoiler. That is, he has a function $f : A \to B$, not necessarily a bijection, that is total, and defines his responses to all the moves by the spoiler. Thus, our reformulation is as follows: in each round i, the spoiler chooses a structure in which to play, say $\mathfrak{A}$. Then the duplicator chooses a total function $f_i : A \to B$. The spoiler then picks an element $a_i \in A$, and the duplicator responds with $f_i(a_i) \in B$.

This presentation of Ehrenfeucht-Fraïssé and bijective games leads to our abstract view of games. The key notion is that in each round, the duplicator has a set of functions (which we call tactics) that will determine his responses to possible moves by the spoiler. As we shall see, this abstract view suffices to capture many games that apply to local logics (but not games for fixed-point or finite-variable logics that are capable of expressing non-local queries). We shall also look at the more general case of structures with parameters $(\mathfrak{A}, \bar{a}_0)$ to capture expressibility by formulae with free variables.

Definition 3.1. An *agreement* $\mathfrak{F}$ assigns to each pair A, B of finite subsets of U a collection

$$\mathfrak{F}(A, B) = \{\mathcal{F}_1(A, B), \dots, \mathcal{F}_m(A, B)\},$$

where each $\mathcal{F}_i(A, B)$ is a nonempty collection of partial functions $f : A \to B$. We call the sets $\mathcal{F}_i(A, B)$ *tactics*.

The $\mathfrak{F}$-*game* on $(\mathfrak{A}, \bar{a}_0)$ and $(\mathfrak{B}, \bar{b}_0)$ is played as follows. Suppose after i rounds the position is $(\bar{a}_0\bar{a}, \bar{b}_0\bar{b})$ (before the game starts, the tuples $\bar{a}, \bar{b}$ are empty). Then, in round $i + 1$:

(1) The spoiler chooses a structure, $\mathfrak{A}$ or $\mathfrak{B}$. Below we present the moves assuming he chose $\mathfrak{A}$; the case of $\mathfrak{B}$ is symmetric.
(2) The duplicator chooses a tactic $\mathcal{F}(A, B) \in \mathfrak{F}(A, B)$.
(3) The spoiler chooses a partial function $f \in \mathcal{F}(A, B)$ and an element $a \in \mathrm{dom}(f)$; the game continues from the position $(\bar{a}_0\bar{a}a,\ \bar{b}_0\bar{b}f(a))$.

The duplicator wins after k-rounds if both $\mathfrak{F}(A, B)$ and $\mathfrak{F}(B, A)$ are non-empty, and the final position defines a partial isomorphism between $(\mathfrak{A}, \bar{a}_0)$ and $(\mathfrak{B}, \bar{b}_0)$. If the duplicator has a winning strategy for the k-round game, we write $(\mathfrak{A}, \bar{a}_0) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b}_0)$.

We now show how some known games can be defined in this setting. In particular, we define four agreements: $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$, $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$, and $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}))$ for an arbitrary unary quantifier $\mathbf{Q}$.

- $\mathfrak{F}(\mathrm{FO})$: as explained above, a tactic is a singleton set $\{f\}$, where $f : A \to B$ is a total function, and $\mathfrak{F}(A, B)$ contains all possible tactics.
- $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$: same as above, except that each tactic is $\{f\}$ where $f : A \to B$ is a bijection (there are no tactics if $|A| \neq |B|$). This is the setting of bijective games.
- $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$: given $A, B \subset U$, a tactic $\mathcal{F}(A, B)$ is a set of maps such that for every $D \subseteq B$, there exists $f \in \mathcal{F}(A, B)$ such that $\mathrm{dom}(f) = A$ and $|f^{-1}(D)| \equiv |D| \pmod p$. Again, $\mathfrak{F}(A, B)$ contains all possible tactics. This corresponds to the game for modulo quantifiers $\mathbf{Q}_p$ [22].

- $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_S))$: given $A, B \subset U$, a tactic $\mathcal{F}(A, B)$ is the union of two sets: $\{f\}$, with f a total function, and a set of maps such that for every $D \subseteq B$ with $|D| \in S$, there exists a function g in this set with $\mathrm{dom}(g) = A$ and $|g^{-1}(D)| \in S$. This corresponds to the game for $\mathrm{FO}(\mathbf{Q}_S)$ [17].

  For multiple unary quantifiers $\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_m}$, tactics in the agreements $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_m}))$ are defined as component-wise unions of tactics from $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_i}))$, for $i \leq m$.

Definition 3.2. Given an agreement $\mathfrak{F}$, we say that the $\mathfrak{F}$-game is a *game for a logic* $\mathcal{L}$, if there exists a partition $\{\mathcal{L}_0, \mathcal{L}_1, \ldots\}$ of the formulae in $\mathcal{L}$ such that for every $k \geq 0$, there exists $k' \geq 0$ with the property that

$$(\mathfrak{A}, \bar{a}_0) \equiv_{k'}^{\mathfrak{F}} (\mathfrak{B}, \bar{b}_0) \text{ implies } \big(\mathfrak{A} \models \varphi(\bar{a}_0) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b}_0)\big), \quad \text{for all } \varphi \in \mathcal{L}_k.$$

If the converse holds as well, that is, for every $k' \geq 0$ there exists $k \geq 0$ such that, $(\mathfrak{A}, \bar{a}_0) \equiv_{k'}^{\mathfrak{F}} (\mathfrak{B}, \bar{b}_0)$, whenever $\mathfrak{A} \models \varphi(\bar{a}_0) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b}_0)$ for every $\varphi \in \mathcal{L}_k$, then we say that the $\mathfrak{F}$-game *captures* $\mathcal{L}$.

All logics considered here have a notion of the quantifier ranks of their formulae, and we shall always associate $\mathcal{L}_k$ with the set of $\mathcal{L}$-formulae of quantifier rank k.

Notice that if $\mathfrak{F}$ is a game for a logic $\mathcal{L}$, and $\mathfrak{F}'$-games capture $\mathcal{L}$, then for every $k \geq 0$ there exists $k' \geq 0$ such that

$$(\mathfrak{A}, \bar{a}) \equiv_{k'}^{\mathfrak{F}} (\mathfrak{B}, \bar{b}) \implies (\mathfrak{A}, \bar{a}) \equiv_{k}^{\mathfrak{F}'} (\mathfrak{B}, \bar{b}).$$

In finite model theory, games are typically used in *inexpressibility* proofs, in which case one only needs the condition that a given game is a game for a logic. In many cases, however, the converse holds too; that is, games completely characterize logics. The following is a reformulation, under our view of games, of standard results on characterizing logics by games [3,14,11,13,22,17,26].

Proposition 3.3. *If $\mathcal{L}$ is one of* FO, $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$, $\mathrm{FO}(\mathbf{Q}_p)$ *or* $\mathrm{FO}(\mathbf{Q}_S)$, *then $\mathfrak{F}(\mathcal{L})$-games are games for $\mathcal{L}$, with $\mathcal{L}_k$ being the set of $\mathcal{L}$-formulae of quantifier rank k. Furthermore, for the cases of* FO, $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$, *and* $\mathrm{FO}(\mathbf{Q}_p)$, *the games capture the corresponding logics.*

4. Locality

Given a σ-structure $\mathfrak{A}$, its *Gaifman graph*, denoted by $G(\mathfrak{A})$, has A as its set of nodes. There is an edge (a_1, a_2) in $G(\mathfrak{A})$ iff there is a relation symbol R in σ such that for some tuple t in the interpretation of this relation in $\mathfrak{A}$, both a_1, a_2 occur in t. By the *distance* $d(a_1, a_2)$ we mean the distance in the Gaifman graph, with $d(a, a) = 0$. If there is no path from a_1 to a_2 in $G(\mathfrak{A})$, then $d(a_1, a_2) = \infty$. We write $d(\bar{a}, \bar{b})$ for the minimum of $d(a, b)$, where $a \in \bar{a}$ and $b \in \bar{b}$.

Let $\mathfrak{A}$ be a σ-structure, and $\bar{a} = (a_1, \ldots, a_m) \in A^m$. The *radius r ball around $\bar{a}$* is the set $B_r^{\mathfrak{A}}(\bar{a}) = \{b \in A \mid d(\bar{a}, b) \leq r\}$. The *$r$-neighbourhood of $\bar{a} = (a_1, \ldots, a_m)$ in $\mathfrak{A}$* is the structure $N_r^{\mathfrak{A}}(\bar{a})$ over vocabulary σ expanded with m constant symbols, where the universe is $B_r^{\mathfrak{A}}(\bar{a})$; the σ-relations are restrictions of the σ-relations in $\mathfrak{A}$ to $B_r^{\mathfrak{A}}(\bar{a})$, and the m additional constants are interpreted as $a_1, \ldots, a_m$. Notice that since we define a neighbourhood around an m-tuple as a structure with additional constant symbols, for any isomorphism h between $N_r^{\mathfrak{A}}(a_1, \ldots, a_m)$ and $N_r^{\mathfrak{B}}(b_1, \ldots, b_m)$, it must be the case that $h(a_i) = b_i$, $1 \leq i \leq m$.

Let $\mathfrak{A}, \mathfrak{B}$ be σ-structures, where σ only contains relation symbols. Let $\bar{a} \in A^m$ and $\bar{b} \in B^m$. We write $(\mathfrak{A}, \bar{a}) \leftrightarrows_d (\mathfrak{B}, \bar{b})$ if there exists a bijection $f : A \to B$ such that

$$N_d^{\mathfrak{A}}(\bar{a}c) \;\cong\; N_d^{\mathfrak{B}}(\bar{b} f(c)), \text{ for every } c \in A.$$

This definition is most commonly used when $m = 0$; then $\mathfrak{A} \leftrightarrows_d \mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ realize the same multiset of isomorphism types of d-neighbourhoods of points. Equivalently, for some bijection $f : A \to B$, we have $N_d^{\mathfrak{A}}(c) \;\cong\; N_d^{\mathfrak{B}}(f(c))$ for all $c \in A$.

We say that a query Q is *Hanf-local*, if there exists a number $d \geq 0$ such that,

$$(\mathfrak{A}, \bar{a}) \leftrightarrows_d (\mathfrak{B}, \bar{b}) \Longrightarrow (\bar{a} \in Q(\mathfrak{A}) \Leftrightarrow \bar{b} \in Q(\mathfrak{B})).$$

This concept was first introduced by Hanf [10] for FO over infinite structures, then modified by [6] to work for sentences over finite models, and further extended in [12] to formulae with free variables.

Gaifman's theorem [8] states that every FO formula $\varphi(\bar{x})$ is equivalent to a Boolean combination of sentences and formulae in which quantification is restricted to $B_r(\bar{x})$, with r determined by φ. In particular, this implies that for every FO formula, we have two numbers, d and k, such that if $\mathfrak{A}$ and $\mathfrak{B}$ agree on all FO sentences of quantifier-rank $\leq k$ and $N_d^{\mathfrak{A}}(\bar{a}) \cong N_d^{\mathfrak{B}}(\bar{b})$, then $\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{B} \models \varphi(\bar{b})$. This concept is normally used when $\mathfrak{A} = \mathfrak{B}$; then it says that a query Q is *Gaifman-local* if there exists a number $d \geq 0$ such that for every structure $\mathfrak{A}$,

$$N_d^{\mathfrak{A}}(\bar{a}_1) \cong N_d^{\mathfrak{A}}(\bar{a}_2) \Longrightarrow (\bar{a}_1 \in Q(\mathfrak{A}) \Leftrightarrow \bar{a}_2 \in Q(\mathfrak{A})).$$

A query Q is *weakly-local* [19] if the above condition holds for disjoint neighbourhoods, that is, there is a number $d \geq 0$ such that for every structure $\mathfrak{A}$,

$$N_d^{\mathfrak{A}}(\bar{a}_1) \cong N_d^{\mathfrak{A}}(\bar{a}_2) \text{ and } B_d^{\mathfrak{A}}(\bar{a}_1) \cap B_d^{\mathfrak{A}}(\bar{a}_2) = \emptyset \implies (\bar{a}_1 \in Q(\mathfrak{A}) \Leftrightarrow \bar{a}_2 \in Q(\mathfrak{A})).$$

The following implications are known [12,19]:

Hanf-local $\Rightarrow$ Gaifman-local $\Rightarrow$ weakly-local.

Examples of logics in which all formulae are Hanf- (and hence Gaifman and weakly) local are all the logics considered so far: FO, FO($\mathbf{Q}_p$), FO($\mathbf{Q}_{\text{PRIME}}$), $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ [8,12,18,21]. There are examples of formulae that are Gaifman- but not Hanf-local [12] and weakly but not Gaifman-local [19].

We now state the definition that relaxes the requirement of Having isomorphisms of neighbourhoods in the usual notions of locality. For $d, \ell \geq 0$, we use the notation $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{d,\ell} (\mathfrak{B}, \bar{b})$ if there exists a bijection $f : A \to B$ such that

$$N_d^{\mathfrak{A}}(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell} N_d^{\mathfrak{B}}(\bar{b} f(c)), \text{ for every } c \in A.$$

Definition 4.1. An agreement $\mathfrak{F}$ is *Hanf-local* if for every $k, m \in \mathbb{N}$, there exist $d, \ell \in \mathbb{N}$ such that for every two structures $\mathfrak{A}$, $\mathfrak{B}$, and tuples $\bar{a} \in A^m$ and $\bar{b} \in B^m$,

$$(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{d,\ell} (\mathfrak{B}, \bar{b}) \implies (\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_{k} (\mathfrak{B}, \bar{b}).$$

We call $\mathfrak{F}$ *Gaifman-local* if for every $k, m \in \mathbb{N}$, there exist $d, \ell \in \mathbb{N}$ such that for every two structures $\mathfrak{A}$, $\mathfrak{B}$, and tuples $\bar{a} \in A^m$ and $\bar{b} \in B^m$,

$$\mathfrak{A} \equiv^{\mathfrak{F}}_{\ell} \mathfrak{B} \text{ and } N_d^{\mathfrak{A}}(\bar{a}) \equiv^{\mathfrak{F}}_{\ell} N_d^{\mathfrak{B}}(\bar{b}) \implies (\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_{k} (\mathfrak{B}, \bar{b}).$$

Finally, we call $\mathfrak{F}$ *weakly-local* if for every $k, m \in \mathbb{N}$, there exist $d, \ell \in \mathbb{N}$ such that for every $\mathfrak{A}$ and $\bar{a}, \bar{b} \in A^m$,

$$N_d^{\mathfrak{A}}(\bar{a}) \equiv^{\mathfrak{F}}_{\ell} N_d^{\mathfrak{A}}(\bar{b}) \text{ and } B_d^{\mathfrak{A}}(\bar{a}) \cap B_d^{\mathfrak{A}}(\bar{b}) = \emptyset \implies (\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_{k} (\mathfrak{A}, \bar{b}).$$

Notice that the new notions of locality are slightly asymmetric with respect to the classic ones: we define locality of agreements instead of locality of queries. However, if the $\mathfrak{F}$-game is a game for a logic $\mathcal{L}$, then proving the locality of the agreement $\mathfrak{F}$ amounts to proving locality for each formula in $\mathcal{L}$.

Also notice that the notion of Gaifman-locality for agreements as defined above is applied over two different structures $\mathfrak{A}$ and $\mathfrak{B}$, which are not necessarily isomorphic, and thus it represents a shift with respect to the classical notion of locality for queries. We chose the notion over different structures because it is more general than the one that is defined over a single structure. In fact, it is not hard to see that if a logic $\mathcal{L}$ is captured by a Gaifman-local agreement $\mathfrak{F}$, then every $\mathcal{L}$-definable query $\varphi(x_1, \ldots, x_m)$ is Gaifman-local in the game-based sense; that is, there exist $r, \ell \in \mathbb{N}$ such that for every $\mathfrak{A}$ and $\bar{a}, \bar{b} \in A^m$,

$$N_r^{\mathfrak{A}}(\bar{a}) \equiv^{\mathfrak{F}}_{\ell} N_r^{\mathfrak{A}}(\bar{b}) \implies \big(\mathfrak{A} \models \varphi(\bar{a}) \Leftrightarrow \mathfrak{A} \models \varphi(\bar{b})\big).$$

Further, as we will see in Section 8, the stronger notion of locality for agreements implies the existence of "local" normal forms for logical formulae, in the style of Gaifman's theorem.

Our main question is the following: *When is a logic local under its games?* Or, more precisely: suppose $\mathfrak{F}$-games are games for a logic $\mathcal{L}$; is $\mathfrak{F}$ Hanf-, Gaifman-, or weakly-local? If a logic is local under its games, we need an assumption *weaker* than isomorphism in order to prove that formulae cannot distinguish some elements of a structure. Consider, for example, the case of Gaifman-locality, applied to one structure $\mathfrak{A}$. Normally, to derive $\varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$, we would need to assume that $N_d(\bar{a}_1) \cong N_d(\bar{a}_2)$ for some appropriate d. But suppose we know that φ comes from a logic that is Gaifman-local under $\mathfrak{F}$-games. If k is such that $(\mathfrak{A}, \bar{a}_1) \equiv^{\mathfrak{F}}_{k} (\mathfrak{A}, \bar{a}_2)$ implies $\varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2)$, then we find

$d, \ell \in \mathbb{N}$ that ensure

$$N_d^{\mathfrak{A}}(\bar{a}_1) \equiv_\ell^{\mathfrak{F}} N_d^{\mathfrak{A}}(\bar{a}_2) \;\Rightarrow\; (\mathfrak{A}, \bar{a}_1) \equiv_k^{\mathfrak{F}} (\mathfrak{A}, \bar{a}_2) \;\Rightarrow\; \mathfrak{A} \models \varphi(\bar{a}_1) \leftrightarrow \varphi(\bar{a}_2).$$

Thus, instead of the isomorphism of neighbourhoods, we have a weaker requirement that they be indistinguishable by the $\mathfrak{F}$-game, in ℓ rounds.

Even though the notion of locality under games is easier to apply, it is harder to analyze than the standard isomorphism-based locality. For example, if a logic $\mathcal{L}$ is local (Hanf-, or Gaifman-, or weakly) under isomorphisms, and $\mathcal{L}'$ is a sub-logic of $\mathcal{L}$, then $\mathcal{L}'$ is local as well. The same, however, is *not* true for game-based locality, as we shall see, as properties of games guaranteeing locality need not be preserved if one passes to weaker games.

5. Basic structural properties

5.1. Admissible agreements

We now look at some most basic properties of agreements that are expected to hold, and that are true in all games corresponding to the logics mentioned so far (and many others as well). Intuitively, they are: (1) the spoiler is free to play any point he wants to; (2) the duplicator can mimic spoiler's moves when they play on the same structure; (3) the games on $(\mathfrak{A}, \mathfrak{B})$ and $(\mathfrak{B}, \mathfrak{C})$ can be composed into a single game on $(\mathfrak{A}, \mathfrak{C})$, and (4) agreements do not depend on a particular choice of elements of U.

From now on, we shall write $\mathcal{F}(A, B) \in \mathfrak{F}$ instead of $\mathcal{F}(A, B) \in \mathfrak{F}(A, B)$.

Definition 5.1. An agreement $\mathfrak{F}$ is said to be *admissible* if the following hold:

(1) For every $\mathcal{F}(A, B) \in \mathfrak{F}$, we have $\bigcup\{\mathrm{dom}(f) \mid f \in \mathcal{F}(A, B)\} = A$ (the spoiler can play any point he wants to);
(2) For every $A \subset U$, there exists $\mathcal{F}(A, A) \in \mathfrak{F}$ such that every $f \in \mathcal{F}(A, A)$ is the identity on $\mathrm{dom}(f)$ (the duplicator can repeat spoiler's moves if they play on the same set);
(3) For every $\mathcal{F}(A, B)$, $\mathcal{F}(B, C) \in \mathfrak{F}$, the composition $\mathcal{F}(A, B) \circ \mathcal{F}(B, C) = \{g \circ f \mid f \in \mathcal{F}(A, B) \text{ and } g \in \mathcal{F}(B, C)\}$ is a tactic in $\mathfrak{F}$ over (A, C) (games compose);
(4) If $\mathcal{F}(A, B)$ is a tactic in $\mathfrak{F}$, and $g : A' \to A$, $h : B \to B'$ are bijections, then $\{h \circ f \circ g \mid f \in \mathcal{F}(A, B)\}$ is a tactic in $\mathfrak{F}$ over (A', B') (agreements do not depend on the choice of elements of U).

It is an easy observation that the agreements $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$, $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))$, $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ are admissible. The next proposition shows that admissible agreements have nice structural properties, or at least, properties that we would expect our games to have.

Proposition 5.2. *Given an admissible agreement $\mathfrak{F}$ and $m, k \geq 0$,*

(a) $\equiv_k^{\mathfrak{F}}$ *is an equivalence relation on structures* $(\mathfrak{A}, \bar{a})$, $\bar{a} \in A^m$*;*
(b) *If $h : \mathfrak{A} \to \mathfrak{B}$ is an isomorphism, then* $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, h(\bar{a}))$.

Proof. (a) Reflexivity is an immediate consequence of Definition 5.1, and symmetry is an immediate consequence of the definition of games. Let $k \geq 0$. We show that if $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$ and $(\mathfrak{B}, \bar{b}) \equiv_k^{\mathfrak{F}} (\mathfrak{C}, \bar{c})$, then $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{C}, \bar{c})$.

Since $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$ and $(\mathfrak{B}, \bar{b}) \equiv_k^{\mathfrak{F}} (\mathfrak{C}, \bar{c})$, there exist tactics $\mathcal{F}_1(A, B)$, $\mathcal{F}_2(B, A)$, $\mathcal{G}_1(B, C)$, $\mathcal{G}_2(C, B) \in \mathfrak{F}$ and, hence, $\mathcal{F}_1(A, B) \circ \mathcal{G}_1(B, C)$ and $\mathcal{G}_2(C, B) \circ \mathcal{F}_2(B, A)$ are in $\mathfrak{F}$. Thus, the $\mathfrak{F}$-game on $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{C}, \bar{c})$ can be played. It is enough to show that the duplicator can always play in such a way that after i rounds, $i \leq k$, if $(a_1, \ldots, a_i)$ and $(c_1, \ldots, c_i)$ are the moves played by spoiler and duplicator on $\mathfrak{A}$ and $\mathfrak{C}$, respectively, then $(\mathfrak{A}, \bar{a}, a_1, \ldots, a_i) \equiv_{k-i} (\mathfrak{C}, \bar{c}, c_1, \ldots, c_i)$.

Assume that $i < k$ moves have been played successfully. By the induction hypothesis, $(\mathfrak{A}, \bar{a}, a_1, \ldots, a_i) \equiv_{k-i} (\mathfrak{C}, \bar{c}, c_1, \ldots, c_i)$. Without loss of generality, assume that in the $i + 1$-th round of the game on $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{C}, \bar{c})$ the spoiler decides to play in $(\mathfrak{C}, \bar{c})$. Then the duplicator picks a tactic $\mathcal{F}(B, A) \in \mathfrak{F}$ as if he has continued playing on $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{B}, \bar{b})$, and also picks a tactic $\mathcal{G}(C, B)$ as if he has continued playing on $(\mathfrak{B}, \bar{b})$ and $(\mathfrak{C}, \bar{c})$. He presents the spoiler with the tactic $\mathcal{G}(C, B) \circ \mathcal{F}(B, A)$. The spoiler makes a move by picking a function $h \in \mathcal{G}(C, B) \circ \mathcal{F}(B, A)$ and an element $c_{i+1} \in \mathrm{dom}(h)$, and the duplicator responds with $a_{i+1} = h(c_{i+1})$. Given that $h \in \mathcal{G}(C, B) \circ \mathcal{F}(B, A)$, there exists $f \in \mathcal{F}(B, A)$, $g \in \mathcal{G}(C, B)$ and $b_{i+1} \in \mathrm{rng}(g) \cap \mathrm{dom}(f)$ such that $h = f \circ g$, $g(c_{i+1}) = b_{i+1}$ and $f(b_{i+1}) = a_{i+1}$. Furthermore, by the way the strategy is defined, and because $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$ and $(\mathfrak{B}, \bar{b}) \equiv_k^{\mathfrak{F}} (\mathfrak{C}, \bar{c})$, it is the case that $(\mathfrak{A}, \bar{a}, a_1 \cdots a_i a_{i+1}) \equiv_{k-(i+1)} (\mathfrak{B}, \bar{b}, b_1 \cdots b_i b_{i+1})$

and $(\mathfrak{B}, \bar{b}, b_1 \cdots b_i b_{i+1}) \equiv_{k-(i+1)} (\mathfrak{C}, \bar{c}, c_1 \cdots c_i c_{i+1})$. Therefore, $(\mathfrak{A}, \bar{a}, a_1 \cdots a_i a_{i+1}) \equiv_{k-(i+1)} (\mathfrak{C}, \bar{c}, c_1 \cdots c_i c_{i+1})$. This concludes the proof of (a).

(b) Assume that $\mathfrak{F}$ is an admissible agreement, and $|A| = |B|$ for two finite sets A, B. Then there is $\mathcal{F}(B, B)$ such that every partial function $f \in \mathcal{F}(B, B)$ is the identity on $\mathrm{dom}(f)$. Therefore, from condition (4) in the definition of admissibility, for every bijection $h : A \to B$, there is a tactic $\mathcal{F}_h(A, B) \in \mathfrak{F}(A, B)$ such that for every $f \in \mathcal{F}_h(A, B)$, $f = h_{\restriction \mathrm{dom}(f)}$. Furthermore, condition (2) in the definition of admissibility implies that there exists $\mathcal{F}(A, A) \in \mathfrak{F}(A, B)$ such that every $f \in \mathcal{F}(A, A)$ is the identity on $\mathrm{dom}(f)$. Using condition (4), we see that if id_A is the identity function on A then $\mathcal{F}_h(A, B) = \{h \circ f \circ \mathsf{id}_A \mid f \in \mathcal{F}(A, A)\} \in \mathfrak{F}(A, B)$ is the desired tactic. Now $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, h(\bar{a}))$ because in each round $i \le k$, the duplicator can choose $\mathcal{F}_h(A, B) \in \mathfrak{F}$. $\square$

In many logics, the equivalence classes of $\equiv_k^{\mathfrak{F}}$ are definable by formulae (they correspond to *types*, or rank-k types, as k typically refers to the quantifier rank). Then definable sets are unions of types. We introduce an abstract notion of definable sets: a set $S \subseteq A^m$ is $(\mathfrak{F}, k)$-*definable* in $\mathfrak{A}$ if it is closed under $\equiv_k^{\mathfrak{F}}$: that is, $\bar{a} \in S$ and $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{A}, \bar{a}_1)$ imply $\bar{a}_1 \in S$. For admissible agreements, definable sets behave in the expected way.

Proposition 5.3. *If $\mathfrak{F}$ is an admissible agreement, then $(\mathfrak{F}, k)$-definable sets are closed under Boolean combinations and Cartesian products; furthermore, the projection $A^{m+1} \to A^m$ applied to an $(\mathfrak{F}, k)$-definable set is an $(\mathfrak{F}, k+1)$-definable set.*

Proof. The closure under Boolean operations is immediate. The closure under the Cartesian product is an immediate consequence of the fact that if $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$, then $(\mathfrak{A}, \bar{a}_1) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b}_1)$ where $\bar{a}_1$ and $\bar{b}_1$ are similar subtuples of $\bar{a}$ and $\bar{b}$.

Let $S \subseteq A^{m+1}$ be an $(\mathfrak{F}, k)$-definable set, and let S' be its image under the projection $A^{m+1} \to A^m$. Let $\bar{a}' = (a_1, \ldots, a_m) \in S'$, and assume that $(\mathfrak{A}, \bar{a}') \equiv_{k+1}^{\mathfrak{F}} (\mathfrak{A}, \bar{b}')$. Then for some $a_{m+1} \in A$, $(\bar{a}', a_{m+1}) \in S$. By condition (1) of admissibility, the spoiler can play $a_{m+1} \in A$, and thus there is $b_{m+1} \in A$ such that $(\mathfrak{A}, \bar{a}', a_{m+1}) \equiv_k^{\mathfrak{F}} (\mathfrak{A}, \bar{b}', b_{m+1})$. Thus, $(\bar{b}', b_{m+1}) \in S$ and $\bar{b}' \in S'$, which proves closure under projection. $\square$

5.2. Basic agreements

To guarantee the locality of agreements, we impose two very mild conditions on $\mathfrak{F}$-games. The first has to do with compositionality. Composition of games is a standard technique that allows one to use $\mathfrak{A} \equiv_k^{\mathfrak{F}} \mathfrak{A}'$ and $\mathfrak{B} \equiv_k^{\mathfrak{F}} \mathfrak{B}'$ to conclude that $\mathcal{H}(\mathfrak{A}, \mathfrak{B}) \equiv_\ell^{\mathfrak{F}} \mathcal{H}(\mathfrak{A}', \mathfrak{B}')$, for some operation $\mathcal{H}$ (see, e.g., [20] for a survey). While in general such compositionality properties depend on the type of game and the operator $\mathcal{H}$, there is one scenario where they almost universally apply: when $\mathcal{H}$ is the disjoint union of structures [20] (in fact, ℓ is usually equal to k in this situation). We want our games to satisfy this property. We use $\sqcup$ for disjoint union of sets and functions.

Definition 5.4. An agreement $\mathfrak{F}$ is *compositional*, if for every two tactics $\mathcal{F}(A, B)$ and $\mathcal{G}(C, D)$ in $\mathfrak{F}$ such that $A \cap C = B \cap D = \emptyset$, the tactic $\mathcal{F}(A, B) \sqcup \mathcal{G}(C, D)$ defined as the set of disjoint unions of partial functions $f : A \to B$ from $\mathcal{F}(A, B)$ and $g : C \to D$ from $\mathcal{G}(C, D)$ is in $\mathfrak{F}$.

The following proposition, which shows that compositional agreements behave in the expected way, is immediate from the definitions.

Proposition 5.5. *Let $\mathfrak{F}$ be a compositional agreement. If $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$ and $(\mathfrak{C}, \bar{c}) \equiv_k^{\mathfrak{F}} (\mathfrak{D}, \bar{d})$, with $A \cap C = B \cap D = \emptyset$, then $(\mathfrak{A}, \bar{a}) \cup (\mathfrak{C}, \bar{c}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b}) \cup (\mathfrak{D}, \bar{d})$.*

The second condition says that if in a game $\mathfrak{A} \equiv_k^{\mathfrak{F}} \mathfrak{B}$, both players' moves restricted to subsets $C \subseteq A$ and $D \subseteq B$, then such a game may be considered as a game on substructures of $\mathfrak{A}$ and $\mathfrak{B}$ generated by C and D, respectively. Again, this condition is true for practically all reasonable games.

We formalize it as follows. We denote the set of all nonempty restrictions of partial functions from $\mathcal{F}(A, B)$ to $C \subseteq A$ by $\mathcal{F}(A, B)|_C$. Consider a tactic $\mathcal{F}(A, B)$, and nonempty sets $C \subseteq A$ and $D \subseteq B$. We say that $\mathcal{F}(A, B)$ is *shrinkable* to (C, D) if $a \in C \Leftrightarrow f(a) \in D$ for every $f \in \mathcal{F}(A, B)$ and $a \in \mathrm{dom}(f)$.

Definition 5.6. An agreement $\mathfrak{F}$ is *shrinkable* if for every $\mathcal{F}(A, B) \in \mathfrak{F}$, and nonempty subsets $C \subseteq A$ and $D \subseteq B$, if $\mathcal{F}(A, B)$ is shrinkable to (C, D), then $\mathcal{F}(A, B)|_C$ is a tactic over (C, D) that belongs to $\mathfrak{F}$.

An admissible $\mathfrak{F}$ is called *basic* if it is both shrinkable and compositional.

Proposition 5.7. *The agreements* $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ *and* $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ *are basic.*

Proof. For $\mathfrak{F}(\mathrm{FO})$ and $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ — by trivial inspection. We now show that $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ is compositional. Let A, B, C, D be such that $A \cap C = B \cap D = \emptyset$. We have to show that for every $\mathcal{F}(A, B), \mathcal{F}(C, D) \in \mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$, $\mathcal{F}(A, B) \sqcup \mathcal{F}(C, D) \in \mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$. Let $B' \cup D' \subseteq B \cup D$, where $B' \subseteq B$, $D' \subseteq D$. There are $f \in \mathcal{F}(A, B)$ and $g \in \mathcal{F}(C, D)$ such that $\mathrm{dom}(f) = A$, $\mathrm{dom}(g) = C$, $|f^{-1}(B')| \equiv |B'| \pmod p$, and $|g^{-1}(D')| \equiv |D'| \pmod p$. Then the disjoint union h of f and g satisfies $\mathrm{dom}(h) = A \cup C$, and $|h^{-1}(B' \cup D')| \equiv |B' \cup D'| \pmod p$.

Next we show that $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ is shrinkable. Assume that $\mathcal{F}(A, B) \in \mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ is shrinkable to (C, D), where $C \subseteq A$, $D \subseteq B$, and $C, D \neq \emptyset$. We show that $\mathcal{F}(A, B)|_C \in \mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$. Since $\mathcal{F}(A, B)$ is shrinkable to (C, D), we know that $f_{\restriction C}$ is a function from C to D, for each $f \in \mathcal{F}(A, B)$. Consider an arbitrary $D' \subseteq D$. There is a function $f : A \to B \in \mathcal{F}(A, B)$ such that $|f^{-1}(D')| \equiv |D'| \pmod p$, and $f^{-1}(D') \subseteq C$. Hence, $f_{\restriction C}$ is a function from C to D such that $|f^{-1}_{\restriction C}(D')| \equiv |D'| \pmod p$. This finishes the proof. $\square$

Notice, on the other hand, that the agreement $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))$ is not compositional (because the sum of two prime numbers is not necessarily a prime), and hence not basic.

5.3. Technical results on distance and shrinkability

The following results and definitions will be used in most proofs in the rest of the the paper. Lemma 5.8 says how far the duplicator can *see* in a game, while Lemma 5.9 states when a game on two structures can be shrunk to a game of its substructures. For the sake of simplicity, in this section we assume that the maximum arity of a predicate is 2. The results presented here can be easily extended for predicates of higher arity.

Lemma 5.8. *Let* $r > 0$ *and* $k \geq \lceil \log r \rceil$. *Consider* σ*-structures* $(\mathfrak{A}, \bar{a})$ *and* $(\mathfrak{B}, \bar{b})$, *where* $\bar{a} = (a_1, \ldots, a_m)$ *and* $\bar{b} = (b_1, \ldots, b_m)$, *and an admissible agreement* $\mathfrak{F}$. *If* $(\mathfrak{A}, \bar{a}c) \equiv^{\mathfrak{F}}_k (\mathfrak{B}, \bar{b}c')$ *and* $d(a_i, c) \leq r$, *for some* $i \in [1, m]$, *then* $d(b_i, c') = d(a_i, c)$.

Proof. By symmetry it suffices to prove $d(b_i, c') \leq d(a_i, c)$. The proof is by induction on r. First assume that $r = 1$ and $(\mathfrak{A}, \bar{a}c) \equiv^{\mathfrak{F}}_k (\mathfrak{B}, \bar{b}c')$. If $d(a_i, c) \leq r$ ($i \in [1, m]$), then $a_i = c$ or there exists a tuple $R(a_i, c)$ in $\mathfrak{A}$. Thus, given that $(\bar{a}c, \bar{b}c')$ is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, $b_i = c'$ or $R(b_i, c')$ is in $\mathfrak{B}$, and, hence, $d(b_i, c') \leq d(a_i, c)$.

Let $r > 1$ and assume that the property holds for every $r' < r$ and that $d(a_i, c) = l \leq r$ ($i \in [1, m]$). Notice that in this case $k \geq \lceil \log r \rceil \geq 1$. Let a' be an element of $\mathfrak{A}$ such that $d(a_i, a') = \lfloor \frac{l}{2} \rfloor$ and $d(a', c) = \lceil \frac{l}{2} \rceil$, and let $\mathcal{F}(A, B) \in \mathfrak{F}$ be a tactic that is chosen by the duplicator in the first round of the game on $(\mathfrak{A}, \bar{a}c)$ and $(\mathfrak{B}, \bar{b}c')$. Since $\mathfrak{F}$ is an admissible agreement, there exists $f \in \mathcal{F}(A, B)$ such that $a' \in \mathrm{dom}(f)$. Assume that the spoiler picks $a' \in \mathrm{dom}(f)$, and let $f(a')$ be the response of the duplicator. Then $(\mathfrak{A}, \bar{a}ca') \equiv^{\mathfrak{F}}_{k-1} (\mathfrak{B}, \bar{b}c' f(a'))$ and, therefore, $(\mathfrak{A}, a_i a') \equiv^{\mathfrak{F}}_{k-1} (\mathfrak{B}, b_i f(a'))$ and $(\mathfrak{A}, ca') \equiv^{\mathfrak{F}}_{k-1} (\mathfrak{B}, c' f(a'))$. Thus, given that $k - 1 \geq \lceil \log \lceil \frac{r}{2} \rceil \rceil \geq \lceil \log \lceil \frac{l}{2} \rceil \rceil \geq \lceil \log \lfloor \frac{l}{2} \rfloor \rceil$, by the induction hypothesis we conclude that $d(b_i, f(a')) \leq \lfloor \frac{l}{2} \rfloor$ and $d(f(a'), c') \leq \lceil \frac{l}{2} \rceil$. This implies that $d(b_i, c') \leq l = d(a_i, c)$. $\square$

Lemma 5.9. *Let* $r \geq 0$, $k \geq 0$, $r' \geq 2r$, $x \in [r, r' - r]$, $k' = k + \lceil \log x \rceil + 1$. *Consider structures* $(\mathfrak{A}, \bar{a})$ *and* $(\mathfrak{B}, \bar{b})$, *and an agreement* $\mathfrak{F}$ *that is admissible and shrinkable:*

1. *If* $\mathfrak{A}'$ *is a substructure of* $\mathfrak{A}$ *such that* $B^{\mathfrak{A}}_{r'}(\bar{a}) \subseteq A'$, *and* $\mathfrak{B}'$ *is a substructure of* $\mathfrak{B}$ *such that* $B^{\mathfrak{B}}_{r'}(\bar{b}) \subseteq B'$, *then* $(\mathfrak{A}', \bar{a}) \equiv^{\mathfrak{F}}_{k'} (\mathfrak{B}', \bar{b})$ *implies that there exists* $\mathcal{F}(B^{\mathfrak{A}}_x(\bar{a}), B^{\mathfrak{B}}_x(\bar{b})) \in \mathfrak{F}$ *such that, for every function* f *that belongs to it and every* $c \in \mathrm{dom}(f)$, $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b} f(c))$.
2. *If* $\mathfrak{A}'$ *is a substructure of* $\mathfrak{A}$ *such that* $B^{\mathfrak{A}}_r(\bar{a}) \subseteq A'$, *and* $\mathfrak{B}'$ *is a substructure of* $\mathfrak{B}$ *such that* $B^{\mathfrak{B}}_r(\bar{b}) \subseteq B'$, *then* $(\mathfrak{A}', \bar{a}) \equiv^{\mathfrak{F}}_{k'-1} (\mathfrak{B}', \bar{b})$ *implies that for every subtuple* $\bar{a}'$ *of* $\bar{a}$, *and* $\bar{b}'$ *the corresponding subtuple of* $\bar{b}$, *it is the case that* $N^{\mathfrak{A}}_r(\bar{a}') \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b}')$.

Proof. We only prove (1); the proof of (2) is very similar. Since $(\mathfrak{A}', \bar{a}) \equiv_{k'}^{\mathfrak{F}} (\mathfrak{B}', \bar{b})$ and $k' \geq 1$, there exists $\mathcal{F}(A', B') \in \mathfrak{F}$ such that, for every $f \in \mathcal{F}(A', B')$ and $c \in B_x^{\mathfrak{A}}(\bar{a}) \cap \mathrm{dom}(f)$,

$$(\mathfrak{A}', \bar{a}c) \equiv_{k'-1}^{\mathfrak{F}} (\mathfrak{B}', \bar{b} f(c)).$$

It follows from Lemma 5.8 that $\mathcal{F}(A', B')$ is shrinkable to $(B_x^{\mathfrak{A}}(\bar{a}),\ B_x^{\mathfrak{B}}(\bar{b}))$. Let $\mathcal{F}(B_x^{\mathfrak{A}}(\bar{a}), B_x^{\mathfrak{B}}(\bar{b})) = \mathcal{F}(A', B')|_{B_x^{\mathfrak{A}}(\bar{a})}$. For an arbitrary $f \in \mathcal{F}(B_x^{\mathfrak{A}}(\bar{a}), B_x^{\mathfrak{B}}(\bar{b}))$ we show how to build a winning duplicator strategy for the k-round $\mathfrak{F}$-game on $N_r^{\mathfrak{A}}(\bar{a}c)$ and $N_r^{\mathfrak{B}}(\bar{b} f(c))$ from the fact that $(\mathfrak{A}', \bar{a}c) \equiv_{k'-1}^{\mathfrak{F}} (\mathfrak{B}', \bar{b} f(c))$. We prove, by induction on the round $i \leq k$, the following claim which implies the result: If $\mathcal{G}_i$ is a winning tactic chosen by the duplicator in round $i \leq k$ of the $(k'-1)$-round $\mathfrak{F}$-game on $(\mathfrak{A}', \bar{a}c)$ and $(\mathfrak{B}', \bar{b} f(c))$ (where $\mathcal{G}_i$ is a tactic from A' to B' if the spoiler decides to play in structure $N_r^{\mathfrak{A}}(\bar{a}c)$ in round i of the $\mathfrak{F}$-game on $N_r^{\mathfrak{A}}(\bar{a}c)$ and $N_r^{\mathfrak{B}}(\bar{b} f(c))$, and from B' to A' otherwise), then – assuming without loss of generality that in round i the spoiler decides to play in $N_r^{\mathfrak{A}}(\bar{a}c)$ – $\mathcal{G}_i$ is shrinkable to $(B_r^{\mathfrak{A}}(\bar{a}c),\ B_r^{\mathfrak{B}}(\bar{b} f(c)))$, and $\mathcal{G}_i|_{B_r^{\mathfrak{A}}(\bar{a}c)}$ can be chosen by the duplicator as a winning tactic in round i of the $\mathfrak{F}$-game on structures $N_r^{\mathfrak{A}}(\bar{a}c)$ and $N_r^{\mathfrak{B}}(\bar{b} f(c))$.

For round 0 there is nothing to prove. Assume that $(a_1, \ldots, a_{i-1})$ and $(b_1, \ldots, b_{i-1})$ are the moves on $N_r^{\mathfrak{A}}(\bar{a}c)$ and $N_r^{\mathfrak{B}}(\bar{b} f(c))$, respectively, in the first $i-1$ rounds of the $\mathfrak{F}$-game on $N_r^{\mathfrak{A}}(\bar{a}c)$ and $N_r^{\mathfrak{B}}(\bar{b} f(c))$. By the induction hypothesis,

$$(N_r^{\mathfrak{A}}(\bar{a}c), a_1 \cdots a_{i-1}) \equiv_{k-i+1}^{\mathfrak{F}} (N_r^{\mathfrak{B}}(\bar{b} f(c)), b_1 \cdots b_{i-1}),$$

and

$$(\mathfrak{A}', \bar{a}c, a_1 \cdots a_{i-1}) \equiv_{k'-i}^{\mathfrak{F}} (\mathfrak{B}', \bar{b} f(c), b_1 \cdots b_{i-1}).$$

Given that $k' - i \geq 1$, the latter says that there exists $\mathcal{G}_i(A', B') \in \mathfrak{F}$ such that for every g in this tactic and $e \in \mathrm{dom}(g)$,

$$(\mathfrak{A}', \bar{a}c, a_1 \cdots a_{i-1}, e) \equiv_{k'-i-1}^{\mathfrak{F}} (\mathfrak{B}', \bar{b} f(c), b_1 \cdots b_{i-1}, g(e)).$$

Since $\mathfrak{F}$ is admissible and $k' - i - 1 \geq \lceil \log r \rceil$, by Lemma 5.8, we deduce that $e \in B_r^{\mathfrak{A}}(\bar{a}c)$ if and only if $g(e) \in B_r^{\mathfrak{B}}(\bar{b} f(c))$. The latter implies, together with the fact that $B_r^{\mathfrak{A}}(\bar{a}c) \subseteq A'$ and $B_r^{\mathfrak{B}}(\bar{b} f(c)) \subseteq B'$, that $\mathcal{G}_i(A', B')$ is shrinkable to $(B_r^{\mathfrak{A}}(\bar{a}c),\ B_r^{\mathfrak{B}}(\bar{b} f(c)))$, and since $\mathfrak{F}$ is shrinkable we conclude that $\mathcal{G}_i(A', B')|_{B_r^{\mathfrak{A}}(\bar{a}c)}$ is in $\mathfrak{F}$, and can be used by the duplicator in the i-th round of the game on $N_r^{\mathfrak{A}}(\bar{a}c)$ and $N_r^{\mathfrak{B}}(\bar{b} f(c))$ to mimic the winning duplicator strategy in the i-th round of the game on $(\mathfrak{A}', \bar{a}c)$ and $(\mathfrak{B}', \bar{b} f(c))$. $\square$

6. Weak locality

We now move to the first locality condition, weak locality. In many applications of locality, at least for proving expressibility bounds, one actually uses weak locality as it is easier to work with disjoint neighbourhoods. While examples of weakly-local formulae violating other notions of locality exist, they are not particularly natural [19].

Recall that an agreement $\mathfrak{F}$ is weakly-local if for every $k, m \geq 0$, there exist $d, \ell \geq 0$ such that for every structure $\mathfrak{A}$ and every $\bar{a}, \bar{b} \in A^m$, if $N_d^{\mathfrak{A}}(\bar{a}) \equiv_{\ell}^{\mathfrak{F}} N_d^{\mathfrak{A}}(\bar{b})$ and the neighbourhoods $N_d^{\mathfrak{A}}(\bar{a})$ and $N_d^{\mathfrak{A}}(\bar{b})$ are disjoint, then $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{A}, \bar{b})$. We define the *weak-locality rank with respect to* $\mathfrak{F}$, denoted by $\mathrm{wlr}_{\mathfrak{F}}(k, m)$, as the minimum d for which the above condition holds. Our main result is as follows.

Theorem 6.1. *Every basic agreement* $\mathfrak{F}$ *is weakly-local. Furthermore,* $\mathrm{wlr}_{\mathfrak{F}}(k, m) = O(2^k)$.

This immediately implies that the agreements $\mathfrak{F}(\mathrm{FO})$, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ and $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_p))$ are weakly-local, and hence FO, $\mathrm{FO}(\mathbf{Q}_p)$, and $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ are weakly-local under their games.

It might be tempting to think that every extension of FO with simple unary generalized quantifiers is weakly local under its games, but we show that this is not the case.

The counterexample is given by the prime quantifier $\mathbf{Q}_{\mathrm{PRIME}}$. It is known [26,17] that for every $\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}})$-formula $\varphi(\bar{x})$ of quantifier rank k, if $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))} (\mathfrak{B}, \bar{b})$, then $\mathfrak{A} \models \varphi(\bar{a})$ iff $\mathfrak{B} \models \varphi(\bar{b})$. Thus, we show that $\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}})$ is not weakly-local under its games by proving the following.

Proposition 6.2. $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))$ *is not weakly-local.*

In the rest of the section we prove these results. We start with Theorem 6.1. Assume that $\mathfrak{F}$ is a basic agreement and that $\bar{a} = (a_1, \dots, a_m)$ and $\bar{b} = (b_1, \dots, b_m)$. To prove the theorem, we show that

Claim 6.3. *If $N^{\mathfrak{A}}_{2^k}(\bar{a}) \equiv^{\mathfrak{F}}_{2k} N^{\mathfrak{A}}_{2^k}(\bar{b})$ and $B^{\mathfrak{A}}_{2^k}(\bar{a}) \cap B^{\mathfrak{A}}_{2^k}(\bar{b}) = \emptyset$, then $(\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_k (\mathfrak{A}, \bar{b})$.*

To prove the claim, we need to introduce some terminology. Assume that i rounds of the $\mathfrak{F}$-game on $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{A}, \bar{b})$ have been played and that $(c_1, \dots, c_i)$, $(e_1, \dots, e_i)$ are the moves of the spoiler and the duplicator on $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{A}, \bar{b})$, respectively. Furthermore, assume that $\bar{x}$ is either $\bar{a}$ or $\bar{b}$. Then we define d_1 as 2^{k-1}, and we define $C_1(\bar{x})$ and $md_1(\bar{x})$ as follows:

$$C_1(\bar{x}) = \begin{cases} \{c_1\} & d(\bar{x}, c_1) \le d_1 \\ \emptyset & \text{otherwise} \end{cases} \qquad md_1(\bar{x}) = \begin{cases} d(\bar{x}, c_1) & C_1(\bar{x}) \ne \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, $C_1(\bar{x})$ contains c_1 if it is "close" to $\bar{x}$. For every $j \in [2, i]$, we define d_j, $C_j(\bar{x})$ and $md_j(\bar{x})$ inductively by considering the j-th move of the $\mathfrak{F}$-game on $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{A}, \bar{b})$ and the values of d_{j-1}, $C_{j-1}(\bar{x})$ and $md_{j-1}(\bar{x})$. More precisely: d_j is $2^{k-j} + \max\{md_{j-1}(\bar{a}), md_{j-1}(\bar{b})\}$ and

$$C_j(\bar{x}) = \begin{cases} C_{j-1}(\bar{x}) \cup \{c_j\} & d(\bar{x}, c_j) \le d_j \\ C_{j-1}(\bar{x}) & \text{otherwise} \end{cases}$$

$$md_j(\bar{x}) = \begin{cases} \max\limits_{c \in C_j(\bar{x})} d(\bar{x}, c) & C_j(\bar{x}) \ne \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Note that $d_j \le 2^k - 1$ for all j and, therefore, $B^{\mathfrak{A}}_{d_j}(\bar{a}) \subseteq B^{\mathfrak{A}}_{2^k}(\bar{a})$ and $B^{\mathfrak{A}}_{d_j}(\bar{b}) \subseteq B^{\mathfrak{A}}_{2^k}(\bar{b})$. We also define $E_j(\bar{x}) = \{e_\ell \mid \ell \in [1, j] \text{ and } d(\bar{x}, e_\ell) \le d_j\}$.

The following claim shows some basic properties of $C_j(\bar{x})$.

Claim 6.4. *For every $j \le i$, the following hold:*

(a) *If $c_j \notin C_j(\bar{a}) \cup C_j(\bar{b})$, then for every $\ell \in [j+1, i]$ we have $d(\bar{a}, c_j) > d_j > d_\ell$ and $d(\bar{b}, c_j) > d_j > d_\ell$.*
(b) *If $c_j \notin C_j(\bar{a})$ (respectively, $c_j \notin C_j(\bar{b})$), then for every $\ell \in [j, i]$ we have that $d(\bar{a}, c_j) > d_\ell$ (respectively, $d(\bar{b}, c_j) > d_\ell$).*
(c) *If $c_j \notin C_j(\bar{a})$ (respectively, $c_j \notin C_j(\bar{b})$), then $c_j \notin C_i(\bar{a})$ (respectively, $c_j \notin C_i(\bar{b})$).*
(d) *$c_j \in C_i(\bar{a})$ (respectively, $c_j \in C_i(\bar{b})$) if and only if $d(\bar{a}, c_j) \le d_i$ (respectively, $d(\bar{b}, c_j) \le d_i$).*

Proof. The claim easily holds for $j = i$, so we assume that $j < i$.
(a) Assume that $c_j \notin C_j(\bar{a}) \cup C_j(\bar{b})$ and that $\ell \in [j+1, i]$. We show that $d(\bar{a}, c_j) > d_j > d_\ell$ (the other case is similar). We have

$$d_\ell \le \max\{md_j(\bar{a}), md_j(\bar{b})\} + \sum_{p=j+1}^{l} 2^{k-p} \le \max\{md_j(\bar{a}), md_j(\bar{b})\} + 2^{k-j} - 1.$$

Since $c_j \notin C_j(\bar{a}) \cup C_j(\bar{b})$, we have $C_{j-1}(\bar{a}) = C_j(\bar{a})$ and $C_{j-1}(\bar{b}) = C_j(\bar{b})$ and, therefore, $md_{j-1}(\bar{a}) = md_j(\bar{a})$ and $md_{j-1}(\bar{b}) = md_j(\bar{b})$. Thus, given that $d_j = \max\{md_{j-1}(\bar{a}), md_{j-1}(\bar{b})\} + 2^{k-j}$, we conclude that $d_\ell \le d_j - 1 < d_j < d(\bar{a}, c_j)$ since $c_j \notin C_j(\bar{a})$.
(b) Assume that $c_j \notin C_j(\bar{a})$ (the proof is similar for the other case). If $\ell = j$, then by the definition of $C_j(\bar{a})$ we have that $d(\bar{a}, c_j) > d_\ell$. Thus, suppose that $\ell \in [j+1, i]$.

To show that $d(\bar{a}, c_j) > d_\ell$, we consider two cases. First, suppose that $c_j \in C_j(\bar{b})$. Then by the definition of $C_j(\bar{b})$ we have that $d(\bar{b}, c_j) \le d_j \le 2^k$ and, hence, $d(\bar{a}, c_j) > 2^k$ since $B^{\mathfrak{A}}_{2^k}(\bar{a}) \cap B^{\mathfrak{A}}_{2^k}(\bar{b}) = \emptyset$. We conclude that $d(\bar{a}, c_j) > d_\ell$ since $d_\ell \le 2^k$. Second, assume that $c_j \notin C_j(\bar{b})$. Then by (a) we conclude that $d(\bar{a}, c_j) > d_\ell$. Part (c) follows immediately from (b).
(d) Let $c_j \in C_i(\bar{a})$. Then $c_j \in C_{i-1}(\bar{a})$ and, therefore, $d(\bar{a}, c_j) \le md_{i-1}(\bar{a})$, which implies $d(\bar{a}, c_j) \le d_i$. If $c_j \notin C_i(\bar{a})$, then $c_j \notin C_j(\bar{a})$, and by (b) we conclude that $d(\bar{a}, c_j) > d_i$. This finishes the proof of Claim 6.4.

Now we are ready to prove Claim 6.3. By induction on the round number i, next we show that the duplicator can play in such a way that after round i, the following conditions hold:

(1) $(N^{\mathfrak{A}}_{2^k}(\bar{a}), \bar{c}_0) \equiv^{\mathfrak{F}}_{2k-i} (N^{\mathfrak{A}}_{2^k}(\bar{b}), \bar{e}_0)$, where $\bar{c}_0$ is the subtuple of $(c_1, \ldots, c_i)$ that contains all elements in $C_i(\bar{a})$, and $\bar{e}_0$ is the corresponding subtuple of $(e_1, \ldots, e_i)$. Note that $\bar{e}_0$ is the subtuple of $(e_1, \ldots, e_i)$ that contains all elements in $E_i(\bar{b})$, and for each $c_j \in \bar{c}_0$ and $j \le i$, we have $d(\bar{a}, c_j) = d(\bar{b}, e_j)$.
(2) $(N^{\mathfrak{A}}_{2^k}(\bar{b}), \bar{c}_1) \equiv^{\mathfrak{F}}_{2k-i} (N^{\mathfrak{A}}_{2^k}(\bar{a}), \bar{e}_1)$, where $\bar{c}_1$ is the subtuple of $(c_1, \ldots, c_i)$ that contains all elements in $C_i(\bar{b})$, and $\bar{e}_1$ is the corresponding subtuple of $(e_1, \ldots, e_i)$. Note that $\bar{e}_1$ is the subtuple of $(e_1, \ldots, e_i)$ that contains all elements in $E_i(\bar{a})$, and for each $c_j \in \bar{c}_1$ and $j \le i$, we have $d(\bar{b}, c_j) = d(\bar{a}, e_j)$.
(3) Let $\bar{c}_2$ be the subtuple of $(c_1, \ldots, c_i)$ that contains all elements not in $C_i(\bar{a}) \cup C_i(\bar{b})$. Then for each $c_j \in \bar{c}_2$ with $j \le i$ we have $c_j = e_j$. Furthermore, if $\bar{e}_2$ is the subtuple of $(e_1, \ldots, e_i)$ corresponding to $\bar{c}_2$, then it is the case that $\bar{e}_2$ is the subtuple of $(e_1, \ldots, e_i)$ that contains all elements not in $E_i(\bar{a}) \cup E_i(\bar{b})$.
(4) $((\bar{a}, c_1, \ldots, c_i), (\bar{b}, e_1, \ldots, e_i))$ is a partial isomorphism between $(\mathfrak{A}, \bar{a})$ and $(\mathfrak{A}, \bar{b})$.

For $i = 0$, there is nothing to prove. Assume that the property holds for $i < k$; we prove it for $i + 1$. Assume without loss of generality that in the $(i + 1)$st round of the game, the spoiler chooses to play on $(\mathfrak{A}, \bar{a})$. Consider $A' = A \setminus (B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b}))$. Because $\mathfrak{F}$ is admissible, if $A' \neq \emptyset$ there is $\mathcal{F}(A', A') \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(A', A')$ and $c \in \mathrm{dom}(f)$, $f(c) = c$. Also, since $(N^{\mathfrak{A}}_{2^k}(\bar{a}), \bar{c}_0) \equiv^{\mathfrak{F}}_{2k-i} (N^{\mathfrak{A}}_{2^k}(\bar{b}), \bar{e}_0)$, there is $\mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{a}), B^{\mathfrak{A}}_{2^k}(\bar{b})) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{a}), B^{\mathfrak{A}}_{2^k}(\bar{b}))$ and $c \in \mathrm{dom}(f)$,

$$(N^{\mathfrak{A}}_{2^k}(\bar{a}), \bar{c}_0, c) \equiv^{\mathfrak{F}}_{2k-(i+1)} (N^{\mathfrak{A}}_{2^k}(\bar{b}), \bar{e}_0, f(c));$$

and since $(N^{\mathfrak{A}}_{2^k}(\bar{b}), \bar{c}_1) \equiv^{\mathfrak{F}}_{2k-i} (N^{\mathfrak{A}}_{2^k}(\bar{a}), \bar{e}_1)$, there is $\mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{b}), B^{\mathfrak{A}}_{2^k}(\bar{a})) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{b}), B^{\mathfrak{A}}_{2^k}(\bar{a}))$ and $c \in \mathrm{dom}(f)$,

$$(N^{\mathfrak{A}}_{2^k}(\bar{b}), \bar{c}_1, c) \equiv^{\mathfrak{F}}_{2k-(i+1)} (N^{\mathfrak{A}}_{2^k}(\bar{a}), \bar{e}_1, f(c)).$$

Using Lemma 5.8, we see that every $f \in \mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{a}), B^{\mathfrak{A}}_{2^k}(\bar{b}))$ and $c \in \mathrm{dom}(f)$, $c \in B^{\mathfrak{A}}_{d_{i+1}}(\bar{a})$ if and only if $f(c) \in B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})$, since $d_{i+1} \le 2^k$. Furthermore, from the fact that $\mathfrak{F}$ is shrinkable, $\mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{a}), B^{\mathfrak{A}}_{2^k}(\bar{b}))|_{B^{\mathfrak{A}}_{d_{i+1}}(\bar{a})} \in \mathfrak{F}$. Similarly, $\mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{b}), B^{\mathfrak{A}}_{2^k}(\bar{a}))|_{B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})} \in \mathfrak{F}$.

Since $\mathfrak{F}$ is compositional and $B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cap B^{\mathfrak{A}}_{d_{i+1}}(\bar{b}) = \emptyset$, there exists $\mathcal{F}(B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b}), B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})) \in \mathfrak{F}$ that corresponds to the disjoint union of functions in $\mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{a}), B^{\mathfrak{A}}_{2^k}(\bar{b}))|_{B^{\mathfrak{A}}_{d_{i+1}}(\bar{a})}$ and functions in $\mathcal{F}(B^{\mathfrak{A}}_{2^k}(\bar{b}), B^{\mathfrak{A}}_{2^k}(\bar{a}))|_{B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})}$. Also, since $\mathfrak{F}$ is compositional and $(B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})) \cap A' = \emptyset$, there is a tactic $\mathcal{F}(A, A) \in \mathfrak{F}$ that corresponds to the disjoint union of functions in $\mathcal{F}(A', A')$ and functions in $\mathcal{F}(B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b}), B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b}))$. We now show that this tactic $\mathcal{F}(A, A)$ provides the strategy for the duplicator. We prove the four conditions of the induction.

(1) By definition of $C_{i+1}(\bar{a})$, we have $C_i(\bar{a}) \subseteq C_{i+1}(\bar{a})$. Furthermore, if $c_j \notin C_i(\bar{a})$ ($j \in [1, i]$), then $c_j \notin C_j(\bar{a})$ and, therefore, by Claim 6.4(c), we conclude that $c_j \notin C_{i+1}(\bar{a})$. Thus, for every $j \in [1, i]$, we have that $c_j \in C_i(\bar{a})$ if and only if $c_j \in C_{i+1}(\bar{a})$. We use this property to show that for every $j \in [1, i]$, $e_j \in E_i(\bar{b})$ if and only if $e_j \in E_{i+1}(\bar{b})$. Let $j \in [1, i]$. First, assume that $e_j \in E_i(\bar{b})$. Then by the induction hypothesis (1), $c_j \in C_i(\bar{a})$. Thus, $c_j \in C_{i+1}(\bar{a})$ and, therefore, by Claim 6.4(d) we have $d(\bar{a}, c_j) \le d_{i+1}$. Thus, given that $d(\bar{a}, c_j) = d(\bar{b}, e_j)$, by the induction hypothesis (1), we conclude that $d(\bar{b}, e_j) \le d_{i+1}$ and, hence, $e_j \in E_{i+1}(\bar{b})$. Second, assume that $e_j \notin E_i(\bar{b})$. Then by hypothesis (1), $c_j \notin C_i(\bar{a})$. To show that $e_j \notin E_{i+1}(\bar{b})$ we consider two cases. If $c_j \in C_i(\bar{b})$, then, from (2), $e_j \in E_i(\bar{a})$ and, therefore, $d(\bar{a}, e_j) \le d_i \le 2^k$. Since $B^{\mathfrak{A}}_{2^k}(\bar{a}) \cap B^{\mathfrak{A}}_{2^k}(\bar{b}) = \emptyset$, we conclude that $d(\bar{b}, e_j) > 2^k \ge d_{i+1}$ and, hence, $e_j \notin E_{i+1}(\bar{b})$. If $c_j \notin C_i(\bar{b})$, then $c_j \notin C_j(\bar{b})$ and, thus, $d(\bar{b}, c_j) > d_{i+1}$ by Claim 6.4(b). Since $c_j = e_j$ by (3), we conclude that $d(\bar{b}, e_j) > d_{i+1}$ and, hence, $e_j \notin E_{i+1}(\bar{b})$. Next we use this to show that if the duplicator uses $\mathcal{F}(A, A)$ as his strategy in the $(i + 1)$st round of the game, then condition (1) holds.

Suppose first that for the $(i + 1)$ round the spoiler chooses $f \in \mathcal{F}(A, A)$ and $c_{i+1} \in \mathrm{dom}(f)$ such that $c_{i+1} \notin B^{\mathfrak{A}}_{d_{i+1}}(\bar{a})$, that is, $c_{i+1} \notin C_{i+1}(\bar{a})$. Then the subtuple of $(c_1, \ldots, c_i, c_{i+1})$ that contains all elements in $C_{i+1}(\bar{a})$ is $\bar{c}_0$. Furthermore, by the way $\mathcal{F}(A, A)$ is defined, $f(c_{i+1}) \notin B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})$, and hence the subtuple of $(e_1, \ldots, e_i, f(c_{i+1}))$ that contains all elements in $E_{i+1}(\bar{b})$ is $\bar{e}_0$. In this case, (1) holds by the induction hypothesis.

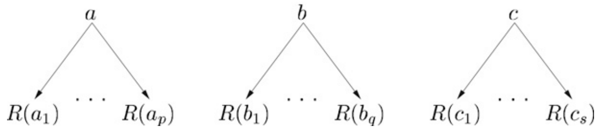

Fig. 2. A structure for proving that FO($\mathbf{Q}_{\text{PRIME}}$) is not weakly-local under its games.

Suppose on the other hand that for round $(i+1)$ the spoiler chooses $f \in F(A, A)$ and $c_{i+1} \in \text{dom}(f)$ such that $c_{i+1} \in B^{\mathfrak{A}}_{d_{i+1}}(\bar{a})$, that is, $c_{i+1} \in C_{i+1}(\bar{a})$. Then $(\bar{c}_0, c_{i+1})$ is the subtuple of $(c_1, \ldots, c_i, c_{i+1})$ that contains all elements in $C_{i+1}(\bar{a})$, and by the definition of $\mathcal{F}(A, A)$, $(N^{\mathfrak{A}}_{2^k}(\bar{a}), \bar{c}_0, c_{i+1}) \equiv^{\mathfrak{F}}_{2k-(i+1)} (N^{\mathfrak{A}}_{2^k}(\bar{b}), \bar{e}_0, f(c_{i+1}))$, and $f(c_{i+1}) \in E_{i+1}(\bar{b})$, and $(\bar{e}_0, f(c_{i+1}))$ is the subtuple of $(e_1, \ldots, e_i, f(c_{i+1}))$ that contains all elements in $E_{i+1}(\bar{b})$. Furthermore, by the induction hypothesis, $d(\bar{a}, c_j) = d(\bar{b}, e_j)$, for each $c_j \in \bar{c}_0$, $j \le i$, and from Lemma 5.8, $d(\bar{a}, c_{j+1}) = d(\bar{b}, f(c_{j+1}))$. This proves (1).

(2) The proof is very similar to the proof for (1).

(3) As in (1) and (2), we conclude that for every $j \in [1, i]$, $c_j \in C_i(\bar{a}) \cup C_i(\bar{b})$ if and only if $c_j \in C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})$ and $e_j \in E_i(\bar{a}) \cup E_i(\bar{b})$ if and only if $e_j \in E_{i+1}(\bar{a}) \cup E_{i+1}(\bar{b})$. Now suppose that for the $(i+1)$st round the spoiler chooses $f \in \mathcal{F}(A, A)$ and $c_{i+1} \in \text{dom}(f)$ such that $c_{i+1} \notin (B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b}))$; that is, $c_{i+1} \notin C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})$. Then $(\bar{c}_2, c_{i+1})$ is the subtuple of $(c_1, \ldots, c_i, c_{i+1})$ that contains all elements not in $C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})$, and by the definition of $\mathcal{F}(A, A)$, $f(c_{i+1}) = c_{i+1}$. Furthermore, $(\bar{e}_2, f(c_{i+1}))$ is the subtuple of $(e_1, \ldots, e_i, f(c_{i+1}))$ that contains all elements not in $E_{i+1}(\bar{a}) \cup E_{i+1}(\bar{b})$, and, by hypothesis (3), for each $c_j \in \bar{c}_2$, $j \le i$, we have $c_j = e_j$. Suppose on the other hand that for the $(i+1)$st round the spoiler chooses $f \in \mathcal{F}(A, A)$ and $c_{i+1} \in \text{dom}(f)$ such that $c_{i+1} \in B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})$, that is, $c \in C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})$. Then $f(c_{i+1}) \in B^{\mathfrak{A}}_{d_{i+1}}(\bar{a}) \cup B^{\mathfrak{A}}_{d_{i+1}}(\bar{b})$. This implies that $\bar{c}_2$ is the subtuple of $(c_1, \ldots, c_i, c_{i+1})$ that contains all elements not in $C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})$, and $\bar{e}_2$ is the subtuple of $(e_1, \ldots, e_i, f(c_{i+1}))$ that contains all elements not in $E_{i+1}(\bar{a}) \cup E_{i+1}(\bar{b})$. In this case, we conclude that (3) holds in the $(i+1)$st round from the induction hypothesis.

(4) Suppose the spoiler chooses $f \in \mathcal{F}(A, A)$ and $c_{i+1} \in \text{dom}(f)$ in round $i+1$. Assume first that for some relation symbol P in the vocabulary, $\mathfrak{A} \models P(c_{j_1}, \ldots, c_{j_p})$, where each $c_{j_\ell} \in (\bar{a}, c_1, \ldots, c_i, c_{i+1})$, $\ell \in [1, p]$.

Since for every $\ell \in [1, k]$, we have $d_\ell \le 2^k - 1$, we conclude that if $c_{\ell_1} \in C_{i+1}(\bar{a})$ and $c_{\ell_2} \in C_{i+1}(\bar{b})$, where $\ell_1, \ell_2 \in [1, i+1]$, then $d(\bar{a}, c_{\ell_1}) < 2^k$ and $d(\bar{b}, c_{\ell_2}) < 2^k$. Thus, given that $B^{\mathfrak{A}}_{2^k}(\bar{a}) \cap B^{\mathfrak{A}}_{2^k}(\bar{b}) = \emptyset$, we conclude that $d(c_{\ell_1}, c_{\ell_2}) > 1$. Similarly, if $c_{\ell_1} \in C_{i+1}(\bar{a})$ and $c_{\ell_2} \notin C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})$, where $\ell_1, \ell_2 \in [1, i+1]$, then by Claim 6.4 we conclude that $d(\bar{a}, c_{\ell_1}) \le d_{i+1}$ and $d(\bar{a}, c_{\ell_2}) > d_{\ell_2} > d_{i+1}$. Thus, $d(c_{\ell_1}, c_{\ell_2}) > 1$. In the same way we conclude that if $c_{\ell_1} \in C_{i+1}(\bar{b})$ and $c_{\ell_2} \notin C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})$, then $d(c_{\ell_1}, c_{\ell_2}) > 1$. Hence, to show that $\mathfrak{A} \models P(e_{j_1}, \ldots, e_{j_p})$ we only have to consider the following cases:

- $\{c_{j_1}, \ldots, c_{j_p}\} \subseteq C_{i+1}(\bar{a})$: then, from condition (1), $\mathfrak{A} \models P(e_{j_1}, \ldots, e_{j_p})$.
- $\{c_{j_1}, \ldots, c_{j_p}\} \subseteq C_{i+1}(\bar{b})$: then, from condition (2), $\mathfrak{A} \models P(e_{j_1}, \ldots, e_{j_p})$.
- $\{c_{j_1}, \ldots, c_{j_p}\} \cap (C_{i+1}(\bar{a}) \cup C_{i+1}(\bar{b})) = \emptyset$: then $\mathfrak{A} \models P(e_{j_1}, \ldots, e_{j_p})$ follows immediately from condition (3).

The proof of the converse, that $\mathfrak{A} \models P(e_{j_1}, \ldots, e_{j_p})$ implies $\mathfrak{A} \models P(c_{j_1}, \ldots, c_{j_p})$, is identical. This finishes the proof of Theorem 6.1. $\square$

Proof of Proposition 6.2. We give a formula $\varphi(x)$ of FO($\mathbf{Q}_{\text{PRIME}}$) such that for every $d, \ell \ge 0$, there is a structure $\mathfrak{A}$ and $a, b \in A$ such that $N^{\mathfrak{A}}_d(a) \equiv^{\mathfrak{F}(\text{FO}(\mathbf{Q}_{\text{PRIME}}))}_{\ell} N^{\mathfrak{A}}_d(b)$, $B^{\mathfrak{A}}_d(a) \cap B^{\mathfrak{A}}_d(b) = \emptyset$, and yet $\mathfrak{A} \models \varphi(a) \wedge \neg\varphi(b)$.

Let σ be a signature of a unary relation R and a binary relation E, and let $d, \ell \ge 0$. Consider the structure $\mathfrak{A}$ whose E-relation is shown in Fig. 2; the relation R is interpreted as the set of all a_i's, b_i's, and c_i's. Let $\varphi(x)$ be $\mathbf{Q}_{\text{PRIME}}\, y(R(y) \wedge \neg E(x, y))$. Notice that for elements a, b, c, their radius-1 neighbourhood equals their radius-d neighbourhood for every $d \ge 1$.

There are infinitely many primes r such that all the numbers $r - i$ $(1 \le i \le \ell)$ are composite. Choose two sufficiently large p, q $(p \ne q)$ from this set so that $N^{\mathfrak{A}}_d(a) \equiv^{\mathfrak{F}(\text{FO}(\mathbf{Q}_{\text{PRIME}}))}_{\ell} N^{\mathfrak{A}}_d(b)$ (notice that d can be taken to be 1, without loss of generality). To see that we can play the ℓ-round $\mathfrak{F}(\text{FO}(\mathbf{Q}_{\text{PRIME}}))$-game on $N^{\mathfrak{A}}_1(a)$ and $N^{\mathfrak{A}}_1(b)$, notice that it suffices to have an ℓ-round winning strategy on sets of cardinalities p and q, for which in turn one has to ensure that for every $i < \ell$, either both $p - i$ and $q - i$ are prime, or both are composite (for if there is a difference, after i

rounds, the spoiler can win in one move). But this is guaranteed by the condition that all $p - i, q - i$ for $i \leq \ell$ are composite.

By Dirichlet's Theorem, the arithmetic progression $np + q$ $(n = 0, 1, \ldots)$ contains an infinite number of primes. Let $n \geq 1$ be such that $np + q$ is a prime and let $s = np$. Then, $\mathfrak{A} \models \varphi(a)$, since $q + s = np + q$ is prime, and $\mathfrak{A} \not\models \varphi(b)$, since $p + s = (n+1)p$ is composite. Thus, the agreement $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{\mathrm{PRIME}}))$ is not weakly-local. □

7. Hanf-locality

Recall that an agreement $\mathfrak{F}$ is Hanf-local, if for every $k, m \geq 0$ there exist $r, \ell \geq 0$ such that, for every two structures $\mathfrak{A}$, $\mathfrak{B}$ and every $\bar{a} \in A^m$ and $\bar{b} \in B^m$, if $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{r,\ell} (\mathfrak{B}, \bar{b})$, then $(\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_{k} (\mathfrak{B}, \bar{b})$. The minimum r for which the above condition holds is called the *Hanf-locality rank with respect to* $\mathfrak{F}$, and is denoted by $\mathrm{hlr}_{\mathfrak{F}}(k, m)$.

Our main result here is the characterization of Hanf-locality for basic agreements (which, by the result of the previous section, possess the simplest of our locality conditions: the weak locality one). We say that an agreement $\mathfrak{F}$ is *bijective* if for every finite $A, B \subset U$, if both $\mathfrak{F}(A, B)$ and $\mathfrak{F}(B, A)$ are non-empty, then $|A| = |B|$. An example of a bijective agreement is $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$. Note that each bijective agreement $\mathfrak{F}$ is equivalent to the agreement $\mathfrak{F}'$ obtained from $\mathfrak{F}$ by removing each nonempty $\mathfrak{F}(A, B)$ such that $|A| \neq |B|$. That is, for every $k \geq 0$, the relations $\equiv^{\mathfrak{F}}_k$ and $\equiv^{\mathfrak{F}'}_k$ are the same. Bijective agreements are then usually identified with games for powerful counting logics, such as $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$. Intuitively, these agreements have the ability to "know" the cardinality of the relevant domain.

Our main result is:

Theorem 7.1. *A basic agreement $\mathfrak{F}$ is Hanf-local if and only if it is bijective.*

In particular, $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ is Hanf-local, and thus $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ is Hanf-local under its games.

We also derive from the Proof of Theorem 7.1 that $\mathrm{hlr}_{\mathfrak{F}}(k, m) = O(3^k)$ for every basic bijective $\mathfrak{F}$.

Non-bijective basic agreements are not Hanf-local, and hence the logics they capture are not Hanf-local under their game. An example of a basic non-bijective agreement is $\mathfrak{F}(\mathrm{FO})$: hence FO is not Hanf-local under Ehrenfeucht-Fraïssé games [23].

The second result of this section addresses the question of the extent to which the results can be pushed for non-basic agreements. That is, suppose a logic $\mathcal{L}$ is Hanf-local under the usual isomorphism-based locality and is captured by a non-basic agreement. Can such a logic be Hanf-local under its games?

We shall give a partially negative answer to this question. Many logics that are known to be Hanf-local under the usual isomorphism-based locality are extensions of FO with unary generalized quantifiers. Agreements that capture such logics are not necessarily basic. We show that such extensions are not Hanf-local under their games, significantly strengthening the negative result in [23].

Theorem 7.2. *No extension of* FO *with a finite collection of simple unary generalized quantifiers is Hanf-local under its games.*

In the rest of the section we prove these results. We start by showing that every basic bijective agreement is Hanf-local. The proof of this follows from three intermediate results presented below.

Lemma 7.3. *If $\mathfrak{F}$ is a basic and bijective agreement, then either:*

(a) *For every nonempty $\mathfrak{F}(A, B)$ it is the case that $|A| \leq |B|$, or*
(b) *For every nonempty $\mathfrak{F}(A, B)$ it is the case that $|A| \geq |B|$.*

Proof. To the contrary, assume that there exist A, B, C and D such that $|A| < |B|$, $|C| > |D|$, and both $\mathfrak{F}(A, B)$ and $\mathfrak{F}(C, D)$ are nonempty. Let $\mathcal{F}(A, B)$ and $\mathcal{F}(C, D)$ be tactics in $\mathfrak{F}(A, B)$ and $\mathfrak{F}(C, D)$, respectively.

Assume that $|C| - |D| = p \geq 1$. Take $|A| \cdot p$ distinct fresh values a_i^j in U, where $i \in [1, |A|]$ and $j \in [1, p]$, and $|B| \cdot p$ distinct fresh values b_i^j in U, where $i \in [1, |B|]$ and $j \in [1, p]$. Let us denote $\{a_i^j \mid j \in [1, p], i \in [1, |A|]\}$ by A_p, and $\{b_i^j \mid j \in [1, p], i \in [1, |B|]\}$ by B_p. Note that $|A_p| = |A| \cdot p$ and $|B_p| = |B| \cdot p$. Then from condition (4) in the definition of $\mathfrak{F}$ being admissible, for every $j \in [1, p]$ there exists a tactic $\mathcal{F}_j(\{a_1^j, \ldots, a_{|A|}^j\}, \{b_1^j, \ldots, b_{|B|}^j\})$ in $\mathfrak{F}$. Since $\mathfrak{F}$ is compositional, we conclude that there exists a tactic $\mathcal{F}(A_p, B_p)$ in $\mathfrak{F}$. In the same way we can show that there exists a tactic $\mathcal{F}(C_q, D_q)$ in $\mathfrak{F}$, where $q = |B| - |A| \geq 1$, C_q is a set of cardinality $|C| \cdot q$, and D_q is a set of cardinality $|D| \cdot q$.

Let us assume without loss of generality that $|B_p| \geq |C_q|$, and $|B_p| - |C_q| = r \geq 0$. Let $E = \{e_1, \ldots, e_r\}$ be r distinct fresh values in U. Then since $\mathfrak{F}$ is admissible, there is a tactic $\mathcal{F}(E, E)$ in $\mathfrak{F}$. Furthermore, since $\mathfrak{F}$ is compositional, there is a tactic $\mathcal{F}(C_q \cup E, D_q \cup E)$ in $\mathfrak{F}$. Notice that

$$|C_q \cup E| = |C_q| + |E| = |C_q| + r = |B_p|$$

and

$$\begin{aligned} |D_q \cup E| = |D_q| + |E| = |D| \cdot q + r &= |D| \cdot q + |B| \cdot p - |C| \cdot q \\ &= |B| \cdot p - p \cdot q \\ &= |B| \cdot p - p \cdot (|B| - |A|) \\ &= |A| \cdot p = |A_p|. \end{aligned}$$

But then from condition (4) in the definition of $\mathfrak{F}$ being admissible, we conclude that there is a tactic $\mathcal{F}(B_p, A_p)$ in $\mathfrak{F}$, which contradicts the fact that $\mathfrak{F}$ is bijective since $\mathfrak{F}(A_p, B_p)$ and $\mathfrak{F}(B_p, A_p)$ are both nonempty and $|A_p| \neq |B_p|$. □

Lemma 7.4. *Let $\mathfrak{F}$ be a basic and bijective agreement, $r \geq 0$, $\ell \geq 0$, $\ell' = \ell + \lceil \log r \rceil$, $\ell'' = \ell' + \lceil \log(2r+1) \rceil + 1$ and $(\mathfrak{A}, \bar{a})$, $(\mathfrak{B}, \bar{b})$ structures over the same signature, with $\bar{a} \in A^m$ and $\bar{b} \in B^m$. If $\mathfrak{A} \leftrightarrows^{\mathfrak{F}}_{r,\ell} \mathfrak{B}$ and $N^{\mathfrak{A}}_{3r+1}(\bar{a}) \equiv^{\mathfrak{F}}_{\ell''} N^{\mathfrak{B}}_{3r+1}(\bar{b})$, then $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{r,\ell} (\mathfrak{B}, \bar{b})$.*

Proof. Given that $N^{\mathfrak{A}}_{3r+1}(\bar{a}) \equiv^{\mathfrak{F}}_{\ell''} N^{\mathfrak{B}}_{3r+1}(\bar{b})$, by Lemma 5.9 there exists a tactic $\mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ in $\mathfrak{F}$ such that for every $f \in \mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ and $c \in \mathrm{dom}(f)$:

$$N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell'} N^{\mathfrak{B}}_r(\bar{b}f(c)).$$

The same argument shows that $\mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{b}), B^{\mathfrak{B}}_{2r+1}(\bar{a}))$ is nonempty, and thus, $|B^{\mathfrak{A}}_{2r+1}(\bar{a})| = |B^{\mathfrak{B}}_{2r+1}(\bar{b})|$ since $\mathfrak{F}$ is bijective.

Let graph(f) be the graph of a function $f : X \to Y$; that is, $\{(x, y) \in X \times Y \mid f(x) = y\}$. Define a relation $\approx$ as the minimal relation that contains

$$\bigcup_{f \in \mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))} \mathrm{graph}(f)$$

and satisfies the following: if $a \approx b'$, $a' \approx b$ and $f(a') = b'$ for some $f \in \mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$, then $a \approx b$. Another way of looking at this relation is the following: $a \approx b$ if there is a sequence $\langle a_0, b_1, a_1, b_2, a_2, \ldots, b_{m-1}, a_{m-1}, b_m \rangle$ where $a_0 = a$, $b_m = b$, and for every i, there are $f, f' \in \mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ such that $b_i = f(a_{i-1}) = f'(a_i)$, $1 \leq i \leq m-1$, and $b_m = f(a_{m-1})$ for some $f \in \mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$. Notice from this that if $a \approx b$, $a' \approx b$, and $a' \approx b'$, then also $a \approx b'$. Notice from the fact that $\mathfrak{F}$ is basic, or more specifically, because $\equiv^{\mathfrak{F}}_k$ is an equivalence relation from Proposition 5.2, that for every $c \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $d \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $c \approx d$, it is the case that $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell'} N^{\mathfrak{B}}_r(\bar{b}d)$.

We use $\approx$ to define relations $\approx_{\mathfrak{A}}$ on $B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $\approx_{\mathfrak{B}}$ on $B^{\mathfrak{B}}_{2r+1}(\bar{b})$. For every $a, a' \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$, $a \approx_{\mathfrak{A}} a'$ if there exists $b \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $a \approx b$ and $a' \approx b$, and for every $b, b' \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$, $b \approx_{\mathfrak{B}} b'$ if there exists $a \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ such that $a \approx b$ and $a \approx b'$. It easily follows from the definition of $\approx$ and the fact that $\mathfrak{F}$ is basic that both $\approx_{\mathfrak{A}}$ and $\approx_{\mathfrak{B}}$ are equivalence relations on $B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $B^{\mathfrak{B}}_{2r+1}(\bar{b})$, respectively. Define $[a]_{\mathfrak{A}}$ and $[b]_{\mathfrak{B}}$ as the equivalence classes of $a \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $b \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$, respectively. We need the following claim.

Claim 7.5. *For every $a \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $b \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $a \approx b$, we have that $|[a]_{\mathfrak{A}}| = |[b]_{\mathfrak{B}}|$.*

Proof. Assume that $a \approx b$, where $a \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $b \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$. If $a_1 \in [a]_{\mathfrak{A}}$, then there exists $b_1 \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $a \approx b_1$ and $a_1 \approx b_1$. Given that $a \approx b$, we have that $a_1 \approx b$. Thus, for every $f \in \mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$, if $f(a_1)$ is defined, then we have that $f(a_1) \in [b]_{\mathfrak{B}}$ since $a_1 \approx f(a_1)$. Moreover, let a_2 be an element of $B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and f a function of $\mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ such that $f(a_2) \in [b]_{\mathfrak{B}}$. Then there exists $a_3 \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ such that $a_3 \approx f(a_2)$ and $a_3 \approx b$. Since $a_2 \approx f(a_2)$, we have that $a_2 \approx b$. Thus, given that $a \approx b$, we conclude that $a_2 \approx_{\mathfrak{A}} a$ and, therefore,

$a_2 \in [a]_{\mathfrak{A}}$. We conclude that $\mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ is shrinkable to $([a]_{\mathfrak{A}}, [b]_{\mathfrak{B}})$ and, hence, there exists a tactic $\mathcal{G}([a]_{\mathfrak{A}}, [b]_{\mathfrak{B}})$ in $\mathfrak{F}$ since this agreement is basic.

Assume first, for the sake of contradiction, that $|[a]_{\mathfrak{A}}| > |[b]_{\mathfrak{B}}|$. Then there exist $c \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $d \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $c \approx d$ and $|[c]_{\mathfrak{A}}| < |[d]_{\mathfrak{B}}|$ (since $|B^{\mathfrak{A}}_{2r+1}(\bar{a})| = |B^{\mathfrak{B}}_{2r+1}(\bar{b})|$). By using the same argument shown above, we conclude that there exists a tactic $\mathcal{G}([c]_{\mathfrak{A}}, [d]_{\mathfrak{B}})$ in $\mathfrak{F}$, which contradicts the fact that $\mathfrak{F}$ is bijective by Lemma 7.3. We arrive at a similar contradiction if we assume that $|[a]_{\mathfrak{A}}| < |[b]_{\mathfrak{B}}|$. We conclude that $|[a]_{\mathfrak{A}}| = |[b]_{\mathfrak{B}}|$. $\square$

It follows from Claim 7.5 that there are partitions of $B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $B^{\mathfrak{B}}_{2r+1}(\bar{b})$ into equivalence classes $[a_1]_{\mathfrak{A}} \dots, [a_m]_{\mathfrak{A}}$ and $[b_1]_{\mathfrak{B}}, \dots, [b_m]_{\mathfrak{B}}$, respectively, such that $a_i \approx b_i$ and $|[a_i]_{\mathfrak{A}}| = |[b_i]_{\mathfrak{B}}|$, for every $i \in [1, m]$. Using the fact that $\mathfrak{F}$ is basic (in particular, condition (4) in the definition of admissibility), we have, for every $i \in [1, m]$, a tactic $\mathcal{G}([a_i]_{\mathfrak{A}}, [b_i]_{\mathfrak{B}})$ in $\mathfrak{F}$ such that $\bigcup_{g \in \mathcal{G}([a_i]_{\mathfrak{A}}, [b_i]_{\mathfrak{B}})} \text{graph}(g)$ is the graph of a bijection from $[a_i]_{\mathfrak{A}}$ to $[b_i]_{\mathfrak{B}}$. Let $\mathcal{G}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ be the disjoint union of all these tactics. Since $\mathfrak{F}$ is a basic agreement, this tactic belongs to it. Furthermore, given $a \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ and $b \in B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $a \approx b$, and given $g \in \mathcal{G}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ for which $g(a)$ is defined, we have $g(a) \in [b]_{\mathfrak{B}}$ and, therefore, there exists $a_1 \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$ such that $a_1 \approx g(a)$ and $a_1 \approx b$. Thus, $a \approx g(a)$. From a previous remark, $a \approx g(a)$ implies

$$N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell'} N^{\mathfrak{B}}_r(\bar{b}g(c)) \tag{1}$$

for every $g \in \mathcal{G}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ and $c \in \text{dom}(g)$. Let h_1 be the bijection whose graph is $\bigcup_{g \in \mathcal{G}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))} \text{graph}(g)$. Since $\ell' = \ell + \lceil \log r \rceil$, by (1) and Lemma 5.9, we obtain that for every $c \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$,

$$N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(\bar{b}h_1(c)) \tag{2}$$

and

$$N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(h_1(c)). \tag{3}$$

Since $\mathfrak{A} \leftrightarrows^{\mathfrak{F}}_{r,\ell} \mathfrak{B}$, there exists a bijection $h_2 : A \to B$ such that $N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(h_2(c))$ for every $c \in A$. By the existence of bijections h_1 and h_2, we conclude that there exists a bijection $h_3 : (A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a})) \to (B \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}))$, such that for every $c \in A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a})$:

$$N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(h_3(c)). \tag{4}$$

Let h be a bijection $h_1 \cup h_3$. Given $c \in A$, if $c \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$, then $h(c) = h_1(c)$ and, therefore, by (2) we know that $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(\bar{b}h_1(c))$. If $c \notin B^{\mathfrak{A}}_{2r+1}(\bar{a})$, then $h(c) = h_3(c)$ and, therefore, $d(\bar{a}, c) > 2r + 1$ and $d(\bar{b}, h(c)) > 2r + 1$. Thus, from (4) and closure under disjoint unions, we conclude that $N^{\mathfrak{A}}_r(\bar{a}) \cup N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(\bar{b}) \cup N^{\mathfrak{B}}_r(h(c))$ and, hence, $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(\bar{b}h(c))$ since $N^{\mathfrak{A}}_r(\bar{a}c) = N^{\mathfrak{A}}_r(\bar{a}) \cup N^{\mathfrak{A}}_r(c)$ and $N^{\mathfrak{B}}_r(\bar{b}h(c)) = N^{\mathfrak{B}}_r(\bar{b}) \cup N^{\mathfrak{B}}_r(h(c))$. We deduce that $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{r,\ell} (\mathfrak{B}, \bar{b})$. This concludes the proof of the lemma. $\square$

Lemma 7.6. *Let $\mathfrak{F}$ be a basic and bijective agreement, $r \geq 0$, $\ell \geq 0$, $\ell' = \ell + \lceil \log r \rceil + \lceil \log(2r + 1) \rceil + 1$. If $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{3r+1,\ell'} (\mathfrak{B}, \bar{b})$, then there exists a bijection $f : A \to B$ such that for every $c \in A$:*

$$(\mathfrak{A}, \bar{a}c) \leftrightarrows^{\mathfrak{F}}_{r,\ell} (\mathfrak{B}, \bar{b}f(c)).$$

Proof. Given that $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{3r+1,\ell'} (\mathfrak{B}, \bar{b})$, there exists a bijection $f : A \to B$ such that for every $c \in A$:

$$N^{\mathfrak{A}}_{3r+1}(\bar{a}c) \equiv^{\mathfrak{F}}_{\ell'} N^{\mathfrak{B}}_{3r+1}(\bar{b}f(c)). \tag{5}$$

Given that $\ell' > \ell + \lceil \log r \rceil$, By Lemma 5.9 we know that for every $c \in A$:

$$N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_r(f(c))$$

and, therefore, $\mathfrak{A} \leftrightarrows^{\mathfrak{F}}_{r,\ell} \mathfrak{B}$. Thus, by (5) and Lemma 7.4 we conclude that for every $c \in A$, it is the case that $(\mathfrak{A}, \bar{a}c) \leftrightarrows^{\mathfrak{F}}_{r,\ell} (\mathfrak{B}, \bar{b}f(c))$. $\square$

We now finish the proof of the Hanf-locality of basic bijective agreements by induction on k. For $k = 0$, assume that $\mathfrak{A}$ and $\mathfrak{B}$ are structures over the same signature and that $\bar{a} \in A^m$ and $\bar{b} \in B^m$, for an arbitrary $m \geq 0$. Furthermore, assume that $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{0,0} (\mathfrak{B}, \bar{b})$. Then by definition of $\leftrightarrows^{\mathfrak{F}}$, there exists a bijection $f : A \to B$ such that for every $c \in A$:

$$N_0^{\mathfrak{A}}(\bar{a}c) \equiv_0^{\mathfrak{F}} N_0^{\mathfrak{B}}(\bar{b}f(c)).$$

In particular, we conclude that $(\bar{a}, \bar{b})$ defines a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Thus, $(\mathfrak{A}, \bar{a}) \equiv_0^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$.

Assume that the property holds for $k \geq 0$: that is, for each $m > 0$, we have $r, \ell \geq 0$ such that for every pair of structures $\mathfrak{A}$ and $\mathfrak{B}$ over the same signature, and for every $\bar{a} \in A^m$ and $\bar{b} \in B^m$, if $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{r,\ell} (\mathfrak{B}, \bar{b})$, then $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$. Next we show that the property holds for $k + 1$.

Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are structures over the same signature and that $\bar{a} \in A^m$ and $\bar{b} \in B^m$. Furthermore, assume that $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{3r+1,\ell'} (\mathfrak{B}, \bar{b})$, where $\ell' = \ell + \lceil \log r \rceil + \lceil \log(2r + 1) \rceil + 1$. Then by Lemma 7.6, there exists a bijection $f : A \to B$ such that for every $c \in A$:

$$(\mathfrak{A}, \bar{a}c) \leftrightarrows^{\mathfrak{F}}_{r,\ell} (\mathfrak{B}, \bar{b}f(c)).$$

Thus, by the induction hypothesis we have that for every $c \in A$:

$$(\mathfrak{A}, \bar{a}c) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b}f(c)). \tag{6}$$

Since $\mathfrak{F}$ is a basic agreement, there exist tactics $\mathcal{F}(A, B)$ and $\mathcal{F}(B, A)$ such that the graph of f is $\bigcup_{g \in \mathcal{G}(A,B)} \text{graph}(g)$ and the graph of the inverse of f is $\bigcup_{g \in \mathcal{G}(B,A)} \text{graph}(g)$. Thus, there exists a winning strategy for the duplicator in the $(k + 1)$-round $\mathfrak{F}$-game on $\mathfrak{A}$ and $\mathfrak{B}$ that initially uses $\mathcal{G}(A, B)$ and $\mathcal{G}(B, A)$, and then uses the strategy given by the induction hypothesis. We conclude that $(\mathfrak{A}, \bar{a}) \equiv_{k+1}^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$. This concludes the proof of Hanf-locality.

We also notice that for every $k, m \geq 0$, we have $\text{hlr}_{\mathfrak{F}}(k, m) = O(3^k)$. Indeed, $\text{hlr}_{\mathfrak{F}}(0, m) = 0$ and $\text{hlr}_{\mathfrak{F}}(k+1, m) \leq 3 \cdot \text{hlr}_{\mathfrak{F}}(k, m + 1) + 1$, and the bound on $\text{hlr}_{\mathfrak{F}}(k, m)$ follows.

We next show the converse, i.e. that a non-bijective basic agreement $\mathfrak{F}$ is not Hanf-local. Assume, for the sake of contradiction, that $\mathfrak{F}$ is basic, non-bijective, and Hanf-local. Then for $k = 2$ and $m = 0$, there exists $r, \ell \geq 0$ such that for every pair of structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A} \leftrightarrows^{\mathfrak{F}}_{d,\ell} \mathfrak{B}$, we have $\mathfrak{A} \equiv_2^{\mathfrak{F}} \mathfrak{B}$.

Since $\mathfrak{F}$ is a non-bijective basic agreement, there exist X and Y such that $|X| < |Y|$ and both $\mathfrak{F}(X, Y)$ and $\mathfrak{F}(Y, X)$ are nonempty. Assume that $|Y| - |X| = p \geq 1$. Then for every $i \in [1, \ell + 1]$, we define sequences $\{X_j^i\}_{j \in \mathbb{N}}$ and $\{Y_j^i\}_{j \in \mathbb{N}}$ as follows. Let $a_1, \ldots, a_i$ be i fresh elements of U. Then

$$X_0^i = X \cup \{a_1, \ldots, a_i\},$$
$$Y_0^i = Y \cup \{a_1, \ldots, a_i\}.$$

Assume that X_j^i and Y_j^i have been defined ($j \geq 0$) and let $b_1, \ldots, b_p$ be p fresh elements. Then

$$X_{j+1}^i = X_j^i \cup \{b_1, \ldots, b_p\},$$
$$Y_{j+1}^i = Y_j^i \cup \{b_1, \ldots, b_p\}.$$

We note that for every $i \in [1, \ell + 1]$ and $j \in \mathbb{N}$, we have that $|X_j^i| = |X| + i + j \cdot p$ and $|Y_j^i| = |X| + i + (j + 1) \cdot p$.

Claim 7.7. *For every $i \in [1, \ell + 1]$ and $j \in \mathbb{N}$, there exist tactics $\mathcal{F}(X_0^i, X_j^i)$ and $\mathcal{F}(X_j^i, X_0^i)$ in $\mathfrak{F}$.*

Proof. Let $i \in [1, \ell + 1]$. First, by induction on $q \in \mathbb{N}$ we show that there exist tactics $\mathcal{F}(X_q^i, Y_q^i)$ and $\mathcal{F}(Y_q^i, X_q^i)$ in $\mathfrak{F}$. For the case of $q = 0$, we note that given that $\mathfrak{F}$ is an admissible agreement, there exists a tactic $\mathcal{F}(\{a_1, \ldots, a_i\}, \{a_1, \ldots, a_i\}) \in \mathfrak{F}$. Since $\mathfrak{F}$ is also compositional and $\{a_1, \ldots, a_i\} \cap X = \emptyset$ and $\{a_1, \ldots, a_i\} \cap Y = \emptyset$, by Proposition 5.5, we conclude that there exist tactics $\mathcal{F}(X_0^i, Y_0^i)$ and $\mathcal{F}(Y_0^i, X_0^i)$ in $\mathfrak{F}$. Assume that the property holds for $q \in \mathbb{N}$. Then X_{q+1}^i and Y_{q+1}^i are constructed by adding p fresh elements to X_q^i and Y_q^i, respectively. Given that by induction hypothesis there exist tactics $\mathcal{F}(X_q^i, Y_q^i)$ and $\mathcal{F}(Y_q^i, X_q^i)$ in $\mathfrak{F}$, and that $\mathfrak{F}$ is admissible and compositional, as in the previous case we conclude that there exist tactics $\mathcal{F}(X_{q+1}^i, Y_{q+1}^i)$ and $\mathcal{F}(Y_{q+1}^i, X_{q+1}^i)$ in $\mathfrak{F}$.

Second, we note that for every $q \in \mathbb{N}$, there exist tactics $\mathcal{F}(Y_q^i, X_{q+1}^i)$ and $\mathcal{F}(X_{q+1}^i, Y_q^i)$ in $\mathfrak{F}$ from the fact that $|Y_q^i| = |X_{q+1}^i|$, and conditions (2) and (4) in the definition of $\mathfrak{F}$ being admissible. Thus, from condition (3) in the definition of $\mathfrak{F}$ being admissible, for every $j \in \mathbb{N}$ there exist tactics $\mathcal{F}(X_0^i, X_j^i)$ and $\mathcal{F}(X_j^i, X_0^i)$ in $\mathfrak{F}$, which proves the claim. $\square$

Assume that $n = |X_0^{\ell+1}| = |X| + \ell + 1$. By Claim 7.7 we know that for every $i \in [1, \ell + 1]$, there exist tactics $\mathcal{F}(X_0^i, X_n^i)$ and $\mathcal{F}(X_n^i, X_0^i)$ in $\mathfrak{F}$. We use this fact to prove the following claim.

Claim 7.8. *Let $\sigma = \{E(\cdot, \cdot)\}$ be a signature, $\mathfrak{A}$ a clique over σ containing n elements, and $\mathfrak{B}$ a clique over σ containing $n \cdot (p + 1)$ elements. Then for every $a \in A$ and $b \in B$, we have $(\mathfrak{A}, a) \equiv_\ell^{\mathfrak{F}} (\mathfrak{B}, b)$.*

Proof. The strategy of the duplicator in the ℓ-round $\mathfrak{F}$-game on $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$ is as follows. Without loss of generality, assume that in the first round the spoiler decides to play in $\mathfrak{A}$. Given that $\mathfrak{F}$ is an admissible agreement, there exists a tactic $\mathcal{F}(\{a\}, \{b\}) \in \mathfrak{F}$. Also, since there exists a tactic $\mathcal{F}(X_0^\ell, X_n^\ell) \in \mathfrak{F}$, and $|A \setminus \{a\}| = n - 1 = |X_0^\ell|$ and $|B \setminus \{b\}| = n \cdot (p + 1) - 1 = |X_n^\ell|$, there exists a tactic $\mathfrak{F}(A \setminus \{a\}, B \setminus \{b\})$ because $\mathfrak{F}$ is admissible. Thus, by the compositionality of $\mathfrak{F}$ and Proposition 5.5 we conclude that there exists a tactic $\mathcal{F}(A, B) \in \mathfrak{F}$ such that $f(a) = b$ for every $f \in \mathcal{F}(A, B)$. The duplicator picks $\mathcal{F}(A, B)$ in the first move. It is easy to see that no matter which element $c_1 \in A$ and function $f \in \mathcal{F}(A, B)$ the spoiler chooses, the resulting position of the game $(a, c_1, b, f(c_1))$ defines a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Let $i \in [1, \ell - 1]$ and $(c_1, \ldots, c_i)$, $(e_1, \ldots, e_i)$ be i moves of the $\mathfrak{F}$-game on $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$, respectively, and assume that $\big((a, c_1, \ldots, c_i), (b, e_1, \ldots, e_i)\big)$ defines a partial isomorphism between $(\mathfrak{A}, a)$ and $(\mathfrak{B}, b)$. Without loss of generality, assume that in the $(i + 1)$ round the spoiler decides to play in $\mathfrak{B}$. Given that $\mathfrak{F}$ is an admissible agreement, there exists a tactic $\mathcal{F}(\{b, e_1, \ldots, e_i\}, \{a, c_1, \ldots, c_i\}) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(\{b, e_1, \ldots, e_i\}, \{a, c_1, \ldots, c_i\})$, we have that $f(b) = a$, and $f(e_j) = c_j$ for every $j \in [1, i]$ for which $f(e_j)$ is defined. Since there exists a tactic $\mathcal{F}(X_n^{\ell-i}, X_0^{\ell-i}) \in \mathfrak{F}$, and $|A \setminus \{a, c_1, \ldots, c_i\}| = n - (i + 1) = |X_0^{\ell-i}|$ and $|B \setminus \{b, e_1, \ldots, e_i\}| = n \cdot (p + 1) - (i + 1) = |X_n^{\ell-i}|$, we have by the admissibility of $\mathfrak{F}$ and Claim 7.7 that there exists a tactic $\mathfrak{F}(B \setminus \{b, e_1, \ldots, e_i\}, A \setminus \{a, c_1, \ldots, c_i\})$ in $\mathfrak{F}$. Thus, by the compositionality of $\mathfrak{F}$ and Proposition 5.5, we conclude that there exists a tactic $\mathcal{F}(B, A) \in \mathfrak{F}$ such that $f(b) = a$, and $f(e_j) = c_j$ for every $f \in \mathcal{F}(B, A)$ and $j \in [1, i]$ for which $f(e_j)$ is defined. The duplicator picks $\mathcal{F}(B, A)$ in the $(i + 1)$ move. It is easy to see that no matter which element $e_{i+1} \in B$ and function $f \in \mathcal{F}(B, A)$ the spoiler chooses, the resulting position of the game $\big((a, c_1, \ldots, c_i, f(e_{i+1})), (b, e_1, \ldots, e_i, e_{i+1})\big)$ defines a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. This concludes the proof of the claim. $\square$

We are ready to show that the assumption that $\mathfrak{F}$ is Hanf-local leads to a contradiction. Let $\sigma = \{E(\cdot, \cdot)\}$ be a signature, $\mathfrak{A}$ a disjoint union of $p+1 \geq 2$ cliques over σ containing n elements each and $\mathfrak{B}$ a clique over σ containing $n \cdot (p+1)$ elements. We note that $\mathfrak{A} \not\equiv_2^{\mathfrak{F}} \mathfrak{B}$, since if the spoiler picks two elements in distinct cliques of $\mathfrak{A}$ then the duplicator cannot respond with two elements of $\mathfrak{B}$ that are not connected by an edge. Thus, if we prove that $\mathfrak{A} \leftrightarrows_{d,\ell}^{\mathfrak{F}} \mathfrak{B}$, then we have a contradiction. We note that if $d = 0$, then $\mathfrak{A} \leftrightarrows_{d,\ell}^{\mathfrak{F}} \mathfrak{B}$ holds trivially. Thus, we assume that $d \geq 1$. But in this case for every $c \in A$, we have that $N_d^{\mathfrak{A}}(c)$ is a clique containing n elements, and for every $e \in B$, we have that $N_d^{\mathfrak{B}}(e)$ is a clique containing $n \cdot (p+1)$ elements. Therefore, by Claim 7.8 we know that $N_d^{\mathfrak{A}}(c) \equiv_\ell^{\mathfrak{F}} N_d^{\mathfrak{B}}(e)$ for every $c \in A$ and $e \in B$. Thus, for every bijection $g : A \to B$ and $c \in A$, we have that $N_d^{\mathfrak{A}}(c) \equiv_\ell^{\mathfrak{F}} N_d^{\mathfrak{B}}(g(c))$ and, hence, $\mathfrak{A} \leftrightarrows_{d,\ell}^{\mathfrak{F}} \mathfrak{B}$. This concludes the proof of Theorem 7.1. $\square$

Proof of Theorem 7.2. We need to show that for every set of simple unary generalized quantifiers $\{\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_p}\}$, the agreement $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_p}))$ is not Hanf-local. As noted in [23], for the case of FO (empty set of simple unary generalized quantifiers) the latter is proved simply by taking G_1 to be the complete graph with $2N$ vertices, and G_2 to be the disjoint union of two complete graphs with N vertices each, where $N \geq \ell$. Any bijection between the nodes of these graphs witnesses $G_1 \leftrightarrows_{r,\ell}^{\mathfrak{F}(\mathrm{FO})} G_2$, and yet G_1 and G_2 disagree on $\exists x \exists y \neg E(x, y)$. The following lemma generalizes this idea. Recall that K_n stands for an n-element clique (complete graph).

Lemma 7.9. *Let $\sigma = \{E(\cdot, \cdot)\}$ be a signature. For every $\ell \in \mathbb{N}$ and $S_1, \ldots, S_p \subseteq \mathbb{N}$, there exist $n, m \geq 1$ such that $n < m$, n divides m and*

$$(K_n, a) \equiv_{\ell}^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} (K_m, b),$$

where a and b are arbitrary elements of K_n and K_m, respectively.

Proof. If $\ell = 0$, then the property trivially holds for $n = 1$ and $m = 2$. Thus, assume that $\ell \geq 1$. For every $i \in [1, p]$, let m_i be defined as:

$$m_i = \begin{cases} (\ell + 1) + \max S_i & S_i \text{ is finite} \\ (2 \cdot \ell + 4)\text{-th element of } S_i & \text{otherwise.} \end{cases}$$

Furthermore, let $q_j = 2^j \cdot \max\{m_1, \ldots, m_p\}$. Then there exist $j_0, j_1 \in \mathbb{N}$ such that $j_0 < j_1$ and for every $i \in [1, p]$ and $j \in [0, \ell]$, we have $(q_{j_0} - j) \in S_i$ if and only if $(q_{j_1} - j) \in S_i$. Let $n = q_{j_0}$ and $m = q_{j_1}$. Assume that the domains of K_n and K_m are A and B, respectively. Thus, we only need to show that $(K_n, a) \equiv_{\ell}^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} (K_m, b)$, where a, b are arbitrary elements of A and B, respectively. For the sake of simplicity, when playing an ℓ-round $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_p}))$-game on (K_n, a) and (K_m, b), we use the game terminology given in [17] instead of using tactics. That is, in each round either the spoiler plays a *point* move or a *quantifier* move. For the *point* move the spoiler chooses a structure, let us say K_n, and an element $a \in A$, and the duplicator responds with $b \in B$. For the *quantifier* move, the spoiler chooses both a structure, let us say K_n, and a quantifier, let us say $\mathbf{Q}_{S_q}$ ($q \in [1, p]$). Then the spoiler chooses $B' \subseteq B$ with $|B'| \in S_q$, and the duplicator responds with $A' \subseteq A$ such that $|A'| \in S_q$. Finally, the spoiler chooses $a \in A$, and the duplicator responds with $b \in B$ such that $a \in A'$ iff $b \in B'$.

The strategy of the duplicator in the ℓ-round $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_p}))$-game on (K_n, a) and (K_m, b) is as follows. Let $i \in [0, \ell - 1]$ and $(c_1, \ldots, c_i)$, $(e_1, \ldots, e_i)$ be the first i moves of the game on (K_n, a) and (K_m, b), respectively, and assume that $\big((a, c_1, \ldots, c_i), (b, e_1, \ldots, e_i)\big)$ defines a partial isomorphism between (K_n, a) and (K_m, b). Notice that if $i = 0$, then only the constants have been played. If in the $(i + 1)$st round the spoiler decides to play a point move, say $e_{i+1} \in B$, then the duplicator responds with $c_{i+1} \in A$ such that $c_{i+1} = a$ iff $e_{i+1} = b$ and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. We note that the duplicator can play in such a way since $|A| \geq \ell + 1$.

Assume now that the spoiler decides to play the quantifier move $\mathbf{Q}_{S_q}$ ($q \in [1, p]$) in the $(i+1)$st round. We consider two cases depending on whether S_q is finite or not. Assume first that S_q is finite. Then the duplicator responds in the following way. Assume without loss of generality that the spoiler picks $B' \subseteq B$ such that $|B'| \in S_q$. Then given that $|B| = q_{j_1} \geq q_0 \geq m_i = (\ell + 1) + \max S_q$, we know that $|B \setminus B'| \geq \ell + 1$. Then the duplicator picks an arbitrary $A' \subseteq A$ such that $|A'| = |B'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. We note that the duplicator can always pick such a set A' since $|A| = q_{j_0} \geq q_0 \geq m_i = (\ell + 1) + \max S_q$ and, thus, $|A \setminus A'| \geq \ell + 1$. If the spoiler picks $c_{i+1} \in A'$ then the duplicator responds with $e_{i+1} \in B'$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $e_{i+1} = e_j$ iff $c_{i+1} = c_j$. Notice that the duplicator can do this, since $|A'| = |B'|$, $a \in A'$ iff $b \in B'$ and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. If the spoiler picks $c_{i+1} \in (A \setminus A')$ then the duplicator responds with $e_{i+1} \in (B \setminus B')$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $e_{i+1} = e_j$ iff $c_{i+1} = c_j$. Notice that the duplicator can do this as well, since $|B \setminus B'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$ and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. In all these cases, it is easy to see that the resulting position $\big((a, c_1, \ldots, c_i, c_{i+1}), (b, e_1, \ldots, e_i, e_{i+1})\big)$ of the game defines a partial isomorphism between (K_n, a) and (K_m, b).

Assume on the other hand that S_q is infinite. Then the strategy of the duplicator is as follows. We consider two cases.

- The spoiler picks $A' \subseteq A$ such that $|A'| \in S_q$. We consider three sub-cases.
  - If $|A'| \leq \ell$, then $|A \setminus A'| \geq \ell + 1$ since $|A| = q_{j_0} \geq q_0 \geq m_i \geq 2 \cdot \ell + 3$. In this case, the duplicator picks an arbitrary $B' \subseteq B$ such that $|B'| = |A'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. We note that the duplicator can pick such a set B' since $|A| < |B|$. If the spoiler picks $e_{i+1} \in B'$, then the duplicator responds with $c_{i+1} \in A'$ such that $c_{i+1} = a$ iff $e_{i+1} = b$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. Notice that the duplicator can do this since $|A'| = |B'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. If the spoiler picks $e_{i+1} \in (B \setminus B')$ then the duplicator responds with $c_{i+1} \in (A \setminus A')$ such that $c_{i+1} = a$ iff $e_{i+1} = b$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. Notice that the duplicator can do this since $|A \setminus A'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$.

- If $|A \setminus A'| \leq \ell$, then $|A'| \geq \ell + 1$ since $|A| \geq 2 \cdot \ell + 3$. In this case, the duplicator picks an arbitrary $B' \subseteq B$ such that $|B'| \in S_q$, $|B \setminus B'| = |A \setminus A'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. We note that the duplicator can pick such a set B' since $|A'| = q_{j_0} - j$ with $j \in [0, \ell]$, and by definition of j_0 and j_1 we have that $(q_{j_0} - j) \in S_q$ iff $(q_{j_1} - j) \in S_q$. If the spoiler picks $e_{i+1} \in B'$ then the duplicator responds with $c_{i+1} \in A'$ such that $c_{i+1} = a$ iff $e_{i+1} = b$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. The duplicator can do this since $|A'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. If the spoiler picks $e_{i+1} \in (B \setminus B')$ then the duplicator responds with $c_{i+1} \in (A \setminus A')$ such that $c_{i+1} = a$ iff $e_{i+1} = b$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. Notice that the duplicator can do this since $|B \setminus B'| = |A \setminus A'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$.
- If $|A'| > \ell$ and $|A \setminus A'| > \ell$, then the duplicator picks an arbitrary $B' \subseteq B$ such that $|B'| = |A'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. We note that the duplicator can pick such a set B' since $|A| < |B|$. If the spoiler picks $e_{i+1} \in B'$ then the duplicator responds with $c_{i+1} \in A'$ such that $c_{i+1} = a$ iff $e_{i+1} = b$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. The duplicator can do this since $|A'| = |B'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. If the spoiler picks $e_{i+1} \in (B \setminus B')$ then the duplicator responds with $c_{i+1} \in (A \setminus A')$ such that $c_{i+1} = a$ iff $e_{i+1} = b$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. Notice that the duplicator can do this since $|A \setminus A'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$.

In all these three cases, it is easy to see that the resulting position of the game $\big((a, c_1, \ldots, c_i, c_{i+1}), (b, e_1, \ldots, e_i, e_{i+1})\big)$ defines a partial isomorphism between (K_n, a) and (K_m, b).

- The spoiler picks $B' \subseteq B$ such that $|B'| \in S_q$. We consider three sub-cases.
  - If $|B'| \leq \ell$, then $|B \setminus B'| \geq \ell + 1$ since $|B| = q_{j_1} \geq q_0 \geq m_i \geq 2 \cdot \ell + 3$. In this case, the duplicator picks an arbitrary $A' \subseteq A$ such that $|A'| = |B'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. We note that the duplicator can pick such a set A' since $|A| \geq 2 \cdot \ell + 3$. If the spoiler picks $c_{i+1} \in A'$, then the duplicator responds with $e_{i+1} \in B'$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $e_{i+1} = e_j$ iff $c_{i+1} = c_j$. The duplicator can do this since $|A'| = |B'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. If the spoiler picks $c_{i+1} \in (A \setminus A')$ then the duplicator responds with $e_{i+1} \in (B \setminus B')$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. Notice that the duplicator can do this since $|B \setminus B'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$.
  - If $|B \setminus B'| \leq \ell$, then $|B'| \geq \ell + 1$ since $|B| \geq 2 \cdot \ell + 3$. In this case, the duplicator picks an arbitrary $A' \subseteq A$ such that $|A'| \in S_q$, $|A \setminus A'| = |B \setminus B'|$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. We note that the duplicator can pick such a set A' since $|B'| = q_{j_1} - j$ with $j \in [0, \ell]$ and by definition of j_0 and j_1 we have that $(q_{j_1} - j) \in S_q$ iff $(q_{j_0} - j) \in S_q$. If the spoiler picks $c_{i+1} \in A'$ then the duplicator responds with $e_{i+1} \in B'$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $e_{i+1} = e_j$ iff $c_{i+1} = c_j$. The duplicator can do this since $|B'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. If the spoiler picks $c_{i+1} \in (A \setminus A')$ then the duplicator responds with $e_{i+1} \in (B \setminus B')$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $c_{i+1} = c_j$ iff $e_{i+1} = e_j$. Notice that the duplicator can do this since $|A \setminus A'| = |B \setminus B'|$, $a \in A'$ iff $b \in B'$ and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$.
  - If $|B'| > \ell$ and $|B \setminus B'| > \ell$, then the duplicator picks an arbitrary $A' \subseteq A$ such that $|A'|$ is equal to the $(\ell + 2)$-th element of S_q, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. We note that the duplicator can pick such a set A' since $|A| = q_{j_0} \geq q_0 \geq m_i$ and m_i is defined as the $(2 \cdot \ell + 4)$-th element of S_q. We also observe that in this case, $|A'| \geq \ell + 1$ and $|A \setminus A'| \geq \ell + 1$. If the spoiler picks $c_{i+1} \in A'$ then the duplicator responds with $e_{i+1} \in B'$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $e_{i+1} = e_j$ iff $c_{i+1} = c_j$. The duplicator can do this since $|B'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$. If the spoiler picks $c_{i+1} \in (A \setminus A')$ then the duplicator responds with $e_{i+1} \in (B \setminus B')$ such that $e_{i+1} = b$ iff $c_{i+1} = a$, and for every $j \in [1, i]$, $e_{i+1} = e_j$ iff $c_{i+1} = c_j$. Notice that the duplicator can do this since $|B \setminus B'| \geq \ell + 1$, $a \in A'$ iff $b \in B'$, and for every $j \in [1, i]$, $c_j \in A'$ iff $e_j \in B'$.

In all these cases, it is easy to see that the resulting position of the game $\big((a, c_1, \ldots, c_i, c_{i+1}), (b, e_1, \ldots, e_i, e_{i+1})\big)$ defines a partial isomorphism between (K_n, a) and (K_m, b). This concludes the proof of the lemma. $\square$

We now conclude the proof of Theorem 7.2. Let $\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_p}$ be a set of simple unary generalized quantifiers such that $\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1}, \ldots, \mathbf{Q}_{S_p}))$ is Hanf-local. Then for $k = 2$ and $m = 0$, there exist $d, \ell \geq 0$ such that for every pair of structures $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A} \leftrightarrows_{d,\ell}^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} \mathfrak{B}$, then $\mathfrak{A} \equiv_2^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} \mathfrak{B}$.

By Lemma 7.9, there exist $n, m \geq 1$ such that $n < m$, n divides m and for every $a \in A$ and $b \in B$:

$$(K_n, a) \equiv_\ell^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} (K_m, b). \tag{7}$$

Let $q = m/n$, and let $\mathfrak{A}$ be a disjoint union of q cliques containing n elements each and let $\mathfrak{B}$ be a clique containing m elements. We note that $\mathfrak{A} \not\equiv_2^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} \mathfrak{B}$ since $\mathfrak{A}$ and $\mathfrak{B}$ disagree on $\exists x \exists y \neg E(x, y)$. Thus, if we prove that $\mathfrak{A} \leftrightarrows_{d,\ell}^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} \mathfrak{B}$, then we have a contradiction. We note that if $d = 0$, then $\mathfrak{A} \leftrightarrows_{d,\ell}^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} \mathfrak{B}$ holds trivially. Thus, we assume that $d \geq 1$. But in this case for every $c \in A$, we have that $N_d^{\mathfrak{A}}(c)$ is a clique containing n elements, and for every $e \in B$, we have that $N_d^{\mathfrak{B}}(e)$ is a clique containing m elements. Therefore, by (7) we know that $N_d^{\mathfrak{A}}(c) \equiv_\ell^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} N_d^{\mathfrak{B}}(e)$ for every $c \in A$ and $e \in B$. Thus, for every bijection $g : A \to B$ and $c \in A$, we have that $N_d^{\mathfrak{A}}(c) \equiv_\ell^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} N_d^{\mathfrak{B}}(g(c))$ and, hence, $\mathfrak{A} \leftrightarrows_{d,\ell}^{\mathfrak{F}(\mathrm{FO}(\mathbf{Q}_{S_1},\ldots,\mathbf{Q}_{S_p}))} \mathfrak{B}$. This concludes the proof of Theorem 7.2. □

8. Gaifman-locality

Recall that $\mathfrak{F}$ is Gaifman-local if for every $k, m \geq 0$ there exist $d, \ell \geq 0$ such that, for every $\mathfrak{A}$ and $\mathfrak{B}$ and every $\bar{a} \in A^m$ and $\bar{b} \in B^m$, we have $(\mathfrak{A}, \bar{a}) \equiv_k^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$ whenever $\mathfrak{A} \equiv_\ell^{\mathfrak{F}} \mathfrak{B}$ and $N_d^{\mathfrak{A}}(\bar{a}) \equiv_\ell^{\mathfrak{F}} N_d^{\mathfrak{B}}(\bar{b})$. The minimum such d is called *Gaifman-locality rank with respect to* $\mathfrak{F}$, and denoted by $\mathrm{lr}_{\mathfrak{F}}(k, m)$.

This is a rather strong notion that implies the existence of normal forms for logical formulae (much in the same way as Gaifman's theorem for FO and its variants imply normal forms for FO formulae [8,23].) Assume that all the relations $\equiv_k^{\mathfrak{F}}$ are of finite index (as they are for FO and several other logics). In that case, every formula is equivalent to a Boolean combination of sentences and formulae evaluated in a neighbourhood of its free variables. More precisely, if $\mathcal{L}$ is a logic captured by an admissible Gaifman-local agreement $\mathfrak{F}$ such that all the relations $\equiv_k^{\mathfrak{F}}$ are of finite index, then for each $\varphi(\bar{x})$ in $\mathcal{L}$, the models $(\mathfrak{A}, \bar{a})$ of $\varphi(\bar{x})$ form a finite union of equivalence classes $\equiv_k^{\mathfrak{F}}$ for some k. If $\mathcal{L}$ is closed under Boolean combinations, this implies that the set of models of $\varphi(\bar{x})$ is a Boolean combination of $\equiv_\ell^{\mathfrak{F}}$ equivalence classes of structures $\mathfrak{A}$ and $\equiv_\ell^{\mathfrak{F}}$ equivalence classes of radius-d neighbourhoods, for some d and ℓ, by Gaifman-locality. It is easy to see that each such an equivalence class is definable by a formula in the logic if it is captured by $\mathfrak{F}$-games. Hence, we obtain the following normal form result.

Proposition 8.1. *Let $\mathcal{L}$ be a logic captured by an admissible Gaifman-local agreement $\mathfrak{F}$, where $\mathfrak{F}$ has the property that for every k, the relations $\equiv_k^{\mathfrak{F}}$ are of finite index. Then, for every $\mathcal{L}$-formula $\varphi(\bar{x})$, one can find a number d, a sequence $\Phi_1, \ldots, \Phi_n$ of $\mathcal{L}$-sentences, a sequence $\varphi_1(\bar{x}), \ldots, \varphi_m(\bar{x})$ of $\mathcal{L}$-formulae, and a Boolean function $\beta : \{0, 1\}^{n+m} \to \{0, 1\}$ such that*

$$\mathfrak{A} \models \varphi(\bar{a}) \quad \Leftrightarrow \quad \beta\Big(\Phi_1(\mathfrak{A}), \ldots, \Phi_n(\mathfrak{A}), \varphi_1(N_d^{\mathfrak{A}}(\bar{a})), \ldots, \varphi_m(N_d^{\mathfrak{A}}(\bar{a}))\Big) = 1$$

where

$$\Phi_i(\mathfrak{A}) = \begin{cases} 1 & \text{if } \mathfrak{A} \models \Phi \\ 0 & \text{if } \mathfrak{A} \models \neg\Phi \end{cases} \quad \text{and} \quad \varphi_j(N_d^{\mathfrak{A}}(\bar{a})) = \begin{cases} 1 & \text{if } N_d^{\mathfrak{A}}(\bar{a}) \models \varphi_j(\bar{a}) \\ 0 & \text{if } N_d^{\mathfrak{A}}(\bar{a}) \models \neg\varphi_j(\bar{a}). \end{cases}$$

Our main goal is to show that FO, $\mathrm{FO}(\mathbf{Q}_p)$, and $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$, are Gaifman-local under their games. For FO, of course, this follows from Gaifman's normal form [8]. As before, we shall state a general condition on agreements that implies Gaifman-locality, and use it to derive Gaifman-locality of the agreements corresponding to these logics.

Definition 8.2.
- Let θ be an equivalence relation on $\mathbb{N}$. We say that an agreement $\mathfrak{F}$ is *uniform with respect to* θ, if for every $A, B \subset U$, $\mathfrak{F}(A, B)$ is the minimal set containing all tactics $\mathcal{F}(A, B)$ such that for each $D \subseteq B$, there is $f \in \mathcal{F}(A, B)$ such that $\mathrm{dom}(f) = A$ and $|D|\theta|f^{-1}(D)|$.
- An agreement $\mathfrak{F}$ is *uniform* if it is uniform with respect to some equivalence relation θ.

- An agreement $\mathfrak{F}$ is *strongly uniform* if it is uniform with respect to an equivalence relation θ which is a congruence with respect to $+$, and the operation $p - q$ for $p > q$, and in addition satisfies the following: there exists $t \geq 0$ such that θ restricted to $[0, t]$ is the identity, and for all $p, q > t$, if $p\theta q$ then $\theta([0, p]) = \theta([0, q])$ (where $\theta(X) = \{s \mid \exists s' \in X\ s'\theta s\}$).

An example of an equivalence relation θ that satisfies the condition for strong uniformity is $n\theta_p m \Leftrightarrow n - m \equiv 0 \pmod p$. It is clear that $\mathfrak{F}(\mathbf{Q}_p)$ is uniform; however, neither $\mathfrak{F}(\mathrm{FO})$ nor $\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ is uniform.

Our main result is as follows.

Theorem 8.3. • *Every basic and strongly-uniform agreement is Gaifman-local.*
- *Let $\mathcal{L}$ be one of* FO, FO($\mathbf{Q}_p$), *or* $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$. *Then there exists an agreement $\mathfrak{G}(\mathcal{L})$ such that:*

(1) *For every $k \geq 0$, the relations $\equiv_k^{\mathfrak{F}(\mathcal{L})}$ and $\equiv_k^{\mathfrak{G}(\mathcal{L})}$ are the same.*
(2) *$\mathfrak{G}(\mathcal{L})$ is basic and strongly uniform.*

We conclude from here that FO, FO($\mathbf{Q}_p$), and $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ are Gaifman-local under their games. The proof also applies to unions of Gaifman-local agreements, which implies that any logic $\mathrm{FO}(\mathbf{Q}_{p_1}, \ldots, \mathbf{Q}_{p_r})$ is also Gaifman-local under its games.

As a corollary to the proof, we derive a bound $O(7^k)$ for $\mathrm{lr}_{\mathfrak{F}}(k, m)$ for a Gaifman-local agreement $\mathfrak{F}$.

There is another way of getting Gaifman-local agreements, given by the result below.

Proposition 8.4. *If $\mathfrak{F}$ is a basic and bijective agreement, then $\mathfrak{F}$ is Gaifman-local, and* $\mathrm{lr}_{\mathfrak{F}}(k, m) \leq 3 \cdot \mathrm{hlr}_{\mathfrak{F}}(k, m) + 1$.

In particular, this shows that the bound on $\mathrm{lr}_{\mathfrak{F}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))}$ can be lowered from $O(7^k)$ to $O(3^k)$. A similar local normal form for the logic $\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})$ was obtained in [16], and the locality rank of the logic is there shown to be $O(2^k)$ (which is the best possible).

In the rest of the section, we prove these results.

Proof of Theorem 8.3. The theorem follows from several intermediate results. Since $\mathfrak{F}$ is basic, all the $\equiv_k^{\mathfrak{F}}$'s are equivalence relations. We write $a \sim_k^r b$ when $N_r^{\mathfrak{A}}(a) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(b)$ (structures $\mathfrak{A}$ and $\mathfrak{B}$ will always be clear from the context).

Lemma 8.5. *Let $\mathfrak{F}$ be a basic agreement. For every $s, k, r \geq 0$, $\ell \geq r$, $\bar{a} = (a_1, \ldots, a_m)$, and $\bar{b} = (b_1, \ldots, b_m)$, there is $k' \geq 0$ such that if $\mathfrak{A} \equiv_{k'}^{\mathfrak{F}} \mathfrak{B}$ and $N_{\ell+r}^{\mathfrak{A}}(\bar{a}) \equiv_{k'}^{\mathfrak{F}} N_{\ell+r}^{\mathfrak{B}}(\bar{b})$, then for every element $e \in B \setminus B_{3\ell}^{\mathfrak{B}}(\bar{b})$, if the set $\{e' \in B \setminus B_{3\ell}^{\mathfrak{B}}(\bar{b}) \mid e \sim_k^r e'\}$ has at least s elements, then the set $\{c \in A \setminus B_{\ell}^{\mathfrak{A}}(\bar{a}) \mid c \sim_k^r e\}$ has at least s elements.*

Proof. We prove the lemma for the case $m = 2$. The case for any $m > 2$ uses exactly the same kind of reasoning. We use $k' = s \cdot (k + 2\lceil \log \ell \rceil + 8)$. (In general, k' is parameterized by m).

Assume without loss of generality that $s > 0$. Fix an arbitrary element $e \in B \setminus B_{3\ell}^{\mathfrak{B}}(\bar{b})$. Consider an element $b \in B \setminus B_{3\ell}^{\mathfrak{B}}(\bar{b})$ such that $b \sim_k^r e$. We now show that there exists an element a in $A \setminus B_{\ell}^{\mathfrak{B}}(\bar{b})$ such that $a \sim_k^r b$.

Given $\mathfrak{A} \equiv_{k+2\lceil \log \ell \rceil + 8}^{\mathfrak{F}} \mathfrak{B}$, there is an element $c_1 \in A$ such that $(\mathfrak{A}, c_1) \equiv_{k+2\lceil \log \ell \rceil + 7}^{\mathfrak{F}} (\mathfrak{B}, b)$. Since $\mathfrak{F}$ is also shrinkable and $\ell \geq r$, from Lemma 5.9 we deduce that $N_r^{\mathfrak{A}}(c_1) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(b)$. If $c_1 \notin B_{\ell}^{\mathfrak{A}}(\bar{a})$ then we set $a = c_1$; otherwise, assume without loss of generality that $c_1 \in B_{\ell}^{\mathfrak{A}}(a_1)$. From $N_{\ell+r}^{\mathfrak{A}}(\bar{a}) \equiv_{k+2\lceil \log \ell \rceil + 5}^{\mathfrak{F}} N_{\ell+r}^{\mathfrak{B}}(\bar{b})$, we deduce that there exists $e_1 \in B_{\ell+r}^{\mathfrak{B}}(\bar{b})$ such that $(N_{\ell+r}^{\mathfrak{A}}(\bar{a}), c_1) \equiv_{k+2\lceil \log \ell \rceil + 4}^{\mathfrak{F}} (N_{\ell+r}^{\mathfrak{B}}(\bar{b}), e_1)$. From Lemma 5.8 and $d(a_1, c_1) \leq \ell$, we deduce that $d(b_1, e_1) \leq \ell$. The latter implies that $d(e_1, b) > 2\ell$. From Lemma 5.9 and the fact that $\ell \geq r$, we obtain that $N_r^{\mathfrak{A}}(\bar{a}c_1) \equiv_{k+\lceil \log r \rceil}^{\mathfrak{F}} N_r^{\mathfrak{B}}(\bar{b}e_1)$, and from Lemma 5.9, we get $N_r^{\mathfrak{A}}(c_1) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(e_1)$.

Now, since $(\mathfrak{A}, c_1) \equiv_{k+2\lceil \log \ell \rceil + 7}^{\mathfrak{F}} (\mathfrak{B}, b)$, there exists an element $c_2 \in A$ such that $(\mathfrak{A}, c_1, c_2) \equiv_{k+2\lceil \log \ell \rceil + 6}^{\mathfrak{F}} (\mathfrak{B}, b, e_1)$, and an $a_1' \in B$ such that $(\mathfrak{A}, c_1, c_2, a_1) \equiv_{k+2\lceil \log \ell \rceil + 5}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1')$. From Lemma 5.9 and $\ell \geq r$, we get $N_r^{\mathfrak{A}}(c_1, c_2, a_1) \equiv_{k+\lceil \log r \rceil}^{\mathfrak{F}} N_r^{\mathfrak{B}}(b, e_1, a_1')$. By Lemma 5.9, $N_r^{\mathfrak{A}}(c_2) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(e_1)$, implying that $N_r^{\mathfrak{A}}(c_2) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(b)$. We show next that $c_2 \notin B_{\ell}^{\mathfrak{A}}(a_1)$. Assume on the contrary that $d(a_1, c_2) \leq \ell$. Since $(\mathfrak{A}, c_1, c_2, a_1) \equiv_{k+2\lceil \log \ell \rceil + 5}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1')$, by Lemma 5.8 we derive that $d(a_1', e_1) \leq \ell$. Also, since $d(a_1, c_1) \leq \ell$, by Lemma 5.8 we obtain that $d(a_1', b) \leq \ell$. We conclude $d(e_1, b) \leq 2\ell$, which is a contradiction.

If $c_2 \notin B_\ell^{\mathfrak{A}}(a_2)$, set $a = c_2$. Otherwise, $c_2 \in B_\ell^{\mathfrak{A}}(a_2)$ and the proof continues as follows. From $(N_{\ell+r}^{\mathfrak{A}}(\bar{a}), c_1) \equiv_{k+2\lceil\log\ell\rceil+4}^{\mathfrak{F}} (N_{\ell+r}^{\mathfrak{B}}(\bar{b}), e_1)$, there exists $e_2 \in B_{\ell+r}^{\mathfrak{B}}(\bar{b})$ such that $(N_{\ell+r}^{\mathfrak{A}}(\bar{a}), c_1, c_2) \equiv_{k+2\lceil\log\ell\rceil+3}^{\mathfrak{F}} (N_{\ell+r}^{\mathfrak{B}}(\bar{b}), e_1, e_2)$. From Lemma 5.8 and $d(a_2, c_2) \leq \ell$, we deduce that $d(b_2, e_2) \leq \ell$. The latter implies that $d(e_2, b) > 2\ell$. Also, from Lemma 5.9 and the fact that $\ell \geq r$, $N_r^{\mathfrak{A}}(\bar{a}c_2) \equiv_{k+\lceil\log r\rceil}^{\mathfrak{F}} N_r^{\mathfrak{B}}(\bar{b}e_2)$, and from Lemma 5.9, $N_r^{\mathfrak{A}}(c_2) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(e_2)$.

Now, since $(\mathfrak{A}, c_1, c_2, a_1) \equiv_{k+2\lceil\log\ell\rceil+5}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1')$, there exists an element $c_3 \in A$ such that $(\mathfrak{A}, c_1, c_2, a_1, c_3) \equiv_{k+2\lceil\log\ell\rceil+4}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1', e_2)$. From Lemma 5.9 and $\ell \geq r$, $N_r^{\mathfrak{A}}(c_1, c_2, a_1, c_3) \equiv_{k+\lceil\log r\rceil}^{\mathfrak{F}} N_r^{\mathfrak{B}}(b, e_1, a_1', e_2)$. By Lemma 5.9, $N_r^{\mathfrak{A}}(c_3) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(e_2)$, implying that $N_r^{\mathfrak{A}}(c_3) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(b)$ (because $\equiv_k^{\mathfrak{F}}$ is transitive). By using an argument similar to the case of c_2, we show that $d(a_1, c_3) \leq \ell$ implies that $d(e_2, b) \leq 2\ell$, which is a contradiction. This shows that $d(a_1, c_3) > \ell$. In the following, we show that $d(a_2, c_3) > \ell$, and hence, that $c_3 \notin B_\ell^{\mathfrak{B}}(\bar{b})$ and we thus can choose a to be c_3.

Assume on the contrary that $d(a_2, c_3) \leq \ell$. Since $(\mathfrak{A}, c_1, c_2, a_1, c_3) \equiv_{k+2\lceil\log\ell\rceil+4}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1', e_2)$, there exists $a_2' \in B$ such that $(\mathfrak{A}, c_1, c_2, a_1, c_3, a_2) \equiv_{k+2\lceil\log\ell\rceil+3}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1', e_2, a_2')$. Furthermore, since $d(a_2, c_3) \leq \ell$ and $d(a_2, c_2) \leq \ell$, we obtain by Lemma 5.8 that $d(e_1, a_2') \leq \ell$ and $d(e_2, a_2') \leq \ell$. From $(N_{\ell+r}^{\mathfrak{A}}(\bar{a}), c_1, c_2) \equiv_{k+2\lceil\log\ell\rceil+3}^{\mathfrak{F}} (N_{\ell+r}^{\mathfrak{B}}(\bar{b}), e_1, e_2)$, we know there is an element $a_2^* \in B_{\ell+r}^{\mathfrak{A}}(\bar{a})$ such that $(N_{\ell+r}^{\mathfrak{A}}(\bar{a}), c_1, c_2, a_2^*) \equiv_{k+2\lceil\log\ell\rceil+2}^{\mathfrak{F}} (N_{\ell+r}^{\mathfrak{B}}(\bar{b}), e_1, e_2, a_2')$, and from Lemma 5.8, $d(e_1, a_2') \leq \ell$ implies that $d(c_1, a_2^*) \leq \ell$, and $d(e_2, a_2') \leq \ell$ implies that $d(c_2, a_2^*) \leq \ell$.

Since $(\mathfrak{A}, c_1, c_2, a_1, c_3, a_2) \equiv_{k+2\lceil\log\ell\rceil+3}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1', e_2, a_2')$, there exists $a_2^{**} \in B$ such that $(\mathfrak{A}, c_1, c_2, a_1, c_3, a_2, a_2^*) \equiv_{k+2\lceil\log\ell\rceil+2}^{\mathfrak{F}} (\mathfrak{B}, b, e_1, a_1', e_2, a_2', a_2^{**})$. By Lemma 5.8 and both $d(c_1, a_2^*) \leq \ell$ and $d(c_2, a_2^*) \leq \ell$, we deduce that $d(b, a_2^{**}) \leq \ell$ and $d(a_2^{**}, e_1) \leq \ell$, implying that $d(b, e_1) \leq 2\ell$, which is a contradiction. Hence, $d(a_2, c_3) > \ell$.

Now, applying the same argument to the remaining $s - 1$ elements in $\{e' \in B \setminus B_{3\ell}^{\mathfrak{B}}(\bar{b}) \mid e' \sim_k^r e\} \setminus \{b\}$, we conclude that $|\{c \in A \setminus B_\ell^{\mathfrak{A}}(\bar{a}) \mid c \sim_k^r e\}| \geq s$. $\square$

Lemma 8.6. *Let $\mathfrak{F}$ be basic and strongly-uniform. For every k and r, there exist $\ell'', \ell' \geq 0$ such that, if*

$$\mathfrak{A} \equiv_{\ell''}^{\mathfrak{F}} \mathfrak{B} \quad \text{and} \quad N_{7r+3}^{\mathfrak{A}}(\bar{a}) \equiv_{\ell'}^{\mathfrak{F}} N_{7r+3}^{\mathfrak{B}}(\bar{b}),$$

where $\bar{a} \in A^m$ and $\bar{b} \in B^m$, then there exists $\mathcal{F}(A, B) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(A, B)$ and $c \in \operatorname{dom}(f)$,

$$N_r^{\mathfrak{A}}(\bar{a}c) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(\bar{b}f(c)).$$

Proof. Let $t \geq 0$ be given by the definition of strong uniformity: that is, θ restricted to $[0, t]$ is the equality. Let

- $\ell'' = \max\{k', k + \lceil\log r\rceil + 1\}$; and
- $\ell' = \max\{k' + \lceil\log(6r+3)\rceil, k + 2\lceil\log(6r+3)\rceil + 2\lceil\log(2r+1)\rceil + t + 1\}$;

where k' is obtained from Lemma 8.5 for the following parameters: $s = \min\{t, 1\}$, and $\ell = 2r + 1$. We show how to construct $\mathcal{F}(A, B)$ using assumptions $\mathfrak{A} \equiv_{\ell''}^{\mathfrak{F}} \mathfrak{B}$ and $N_{7r+3}^{\mathfrak{A}}(\bar{a}) \equiv_{\ell'}^{\mathfrak{F}} N_{7r+3}^{\mathfrak{B}}(\bar{b})$. Let $D \subseteq B$. We need to construct a function $f : A \to B$ such that $|D|\theta|f^{-1}(D)|$, and for every $c \in A$, $N_r^{\mathfrak{A}}(\bar{a}c) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(\bar{b}f(c))$.

Let $D_0 = D \cap N_{2r+1}^{\mathfrak{B}}(\bar{b})$. Since $\mathfrak{F}$ is basic and $\ell' \geq k + \lceil\log(2r+1)\rceil + 1$, by Lemma 5.9 there exists $\mathcal{F}(B_{2r+1}^{\mathfrak{A}}(\bar{a}), B_{2r+1}^{\mathfrak{B}}(\bar{b})) \in \mathfrak{F}$ such that for every function f in it and every $c \in \operatorname{dom}(f)$, $N_r^{\mathfrak{A}}(\bar{a}c) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(\bar{b}f(c))$. In particular, there is $f_0 \in \mathcal{F}(B_{2r+1}^{\mathfrak{A}}(\bar{a}), B_{2r+1}^{\mathfrak{B}}(\bar{b}))$ such that $\operatorname{dom}(f_0) = B_{2r+1}^{\mathfrak{A}}(\bar{a})$, $|D_0|\theta|f_0^{-1}(D_0)|$, and for every $c \in \operatorname{dom}(f_0)$, $N_r^{\mathfrak{A}}(\bar{a}c) \equiv_k^{\mathfrak{F}} N_r^{\mathfrak{B}}(\bar{b}f_0(c))$.

Next, for every $e \in D \setminus B_{2r+1}^{\mathfrak{B}}(\bar{b})$, define its $\sim_k^r$ equivalence class restricted to $D \setminus B_{2r+1}^{\mathfrak{B}}(\bar{b})$, that is, $[e] = \{e' \in D \setminus B_{2r+1}^{\mathfrak{B}}(\bar{b}) \mid e' \sim_k^r e\}$. Let E be a set of representatives of these equivalence classes.

Define $C' = \bigcup_{e \in E}[c]^e$, where for each $e \in E$, the set $[c]^e$ is defined as follows:

- If there is no $e' \in B \setminus (D \cup B_{2r+1}^{\mathfrak{B}}(\bar{b}))$ such that $e' \sim_k^r e$, then $[c]^e = \{c \in A \setminus B_{2r+1}^{\mathfrak{A}}(\bar{a}) \mid c \sim_k^r e\}$.
- Otherwise we choose $[c]^e$ to be a subset of $\{c \in A \setminus B_{2r+1}^{\mathfrak{A}}(\bar{a}) \mid c \sim_k^r e\}$ such that $|[c]^e|\theta|[e]|$.

We need to show that this is well-defined. This is done in the following claim.

Claim 8.7. *Let $\mathfrak{A} \equiv^{\mathfrak{F}}_{\ell''} \mathfrak{B}$ and $N^{\mathfrak{A}}_{7r+1}(\bar{a}) \equiv^{\mathfrak{F}}_{\ell'} N^{\mathfrak{B}}_{7r+1}(\bar{b})$. Then the following hold for every $e \in E$:*

(1) $|\{e' \in B \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\}| \;\theta\; |\{c \in A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}|$.
(2) *If* $|\{e' \in B \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\}| \geq t$, *then* $|\{c \in A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}| \geq t$.

Proof. (1) We first prove that $[c]^e \neq \emptyset$. Assume that $e \in B^{\mathfrak{B}}_{6r+3}(\bar{b})$. Since $N^{\mathfrak{A}}_{7r+3}(\bar{a}) \equiv^{\mathfrak{F}}_{k+2\lceil\log(6r+3)\rceil+1} N^{\mathfrak{B}}_{7r+3}(\bar{b})$ and $\mathfrak{F}$ is admissible, from Lemma 5.9 there exists a function $h : B^{\mathfrak{B}}_{6r+3}(\bar{b}) \to B^{\mathfrak{A}}_{6r+3}(\bar{a})$ such that $N^{\mathfrak{A}}_r(\bar{a}h(e)) \equiv^{\mathfrak{F}}_{k+\lceil\log(6r+3)\rceil} N^{\mathfrak{B}}_r(\bar{b}e)$. From Lemma 5.8, $h(e) \notin B^{\mathfrak{B}}_{2r+1}(\bar{b})$. Also, from Lemma 5.9, $h(e) \sim^r_k e$. Hence, $h(e) \in [c]^e$.

Now assume $e \notin B^{\mathfrak{B}}_{6r+3}(\bar{b})$. Then, from Lemma 5.9 and the facts that $N^{\mathfrak{A}}_{7r+3}(\bar{a}) \equiv^{\mathfrak{F}}_{k'+\lceil\log(6r+3)\rceil} N^{\mathfrak{B}}_{7r+3}(\bar{b})$ and that $\mathfrak{F}$ is basic, $N^{\mathfrak{A}}_{3r+1}(\bar{a}) \equiv^{\mathfrak{F}}_{k'} N^{\mathfrak{B}}_{3r+1}(\bar{b})$. From Lemma 8.5 and $\mathfrak{A} \equiv_{k'} \mathfrak{B}$, we have

$$|\{c \in A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}| \geq 1.$$

Now, since $\mathfrak{F}$ is strongly uniform, the claim follows from the following statements:

$$|\{e' \in B \mid e' \sim^r_k e\}| \;\theta\; |\{c \in A \mid c \sim^r_k e\}| \tag{8}$$
$$|\{e' \in B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\}| \;\theta\; |\{c \in B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}|. \tag{9}$$

We now proceed to prove these statements. For (8), let $S = \{e' \in B \mid e \sim^r_k e'\}$. Since $\mathfrak{F}$ is uniform and $\ell'' \geq k + \lceil\log r\rceil + 1$, there is $\mathcal{F}(A, B) \in \mathfrak{F}$ and a function $h : A \to B \in \mathcal{F}(A, B)$ such that, $|\{c \in A \mid h(c) \in S\}| \theta |S|$, and for every $c \in A$, $(\mathfrak{A}, c) \equiv^{\mathfrak{F}}_{k+\lceil\log r\rceil} (\mathfrak{B}, h(c))$. By Lemma 5.9, $N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(h(c))$. Hence,

$$\{c \in A \mid c \sim^r_k e\} = \{c \in A \mid h(c) \in S\}.$$

For (9), let $S_0 = \{e' \in B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\}$. From Lemma 5.9 and the fact that $\mathfrak{F}$ is basic, there exists $\mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b})) \in \mathfrak{F}$ such that for every function f in it and every $c \in \text{dom}(f)$, $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{k+\lceil\log r\rceil} N^{\mathfrak{B}}_r(\bar{b}f(c))$ (because $N^{\mathfrak{A}}_{7r+3}(\bar{a}) \equiv^{\mathfrak{F}}_{k+2\lceil\log(2r+1)\rceil+1} N^{\mathfrak{B}}_{7r+3}(\bar{b})$). In particular, there is $f' \in \mathcal{F}(B^{\mathfrak{A}}_{2r+1}(\bar{a}), B^{\mathfrak{B}}_{2r+1}(\bar{b}))$ such that $\text{dom}(f') = B^{\mathfrak{A}}_{2r+1}(\bar{a})$, and $|\{c \in B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid f'(c) \in S_0\}| \theta |S_0|$, and for every $c \in \text{dom}(f')$, $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{k+\lceil\log r\rceil} N^{\mathfrak{B}}_r(\bar{b}f'(c))$. From the latter and Lemma 5.9, we deduce that $N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(f'(c))$. This implies that

$$\{c \in B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\} = \{c \in B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid f'(c) \in S_0\}.$$

(2) For the proof of the second item, assume to the contrary that $|\{c \in A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}| = t' < t$, and $|\{e' \in B \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\}| \geq t$. We shall then derive (to contradict the assumption):

$$|\{c \in B^{\mathfrak{A}}_{6r+3}(\bar{a}) \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}| = |\{e' \in B^{\mathfrak{B}}_{6r+3}(\bar{b}) \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\} \quad \text{and}$$
$$|\{c \in A \setminus B^{\mathfrak{A}}_{6r+3}(\bar{a}) \mid c \sim^r_k e\}| = |\{e' \in B \setminus B^{\mathfrak{A}}_{6r+3}(\bar{b}) \mid e' \sim^r_k e\}|.$$

Next we show that both equalities hold.

- Using $N^{\mathfrak{A}}_{7r+3}(\bar{a}) \equiv^{\mathfrak{F}}_{k+\lceil\log(6r+3)\rceil+2\lceil\log r\rceil+t} N^{\mathfrak{B}}_{7r+3}(\bar{b})$ and Lemma 5.9, we derive

$$\max\big(t,\ |\{c \in B^{\mathfrak{A}}_{6r+3}(\bar{a}) \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}|\big) = \max\big(t,\ |\{e' \in B^{\mathfrak{B}}_{6r+3}(\bar{b}) \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\}|\big).$$

  Since $|\{c \in B^{\mathfrak{A}}_{6r+3}(\bar{a}) \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}| \leq t' < t$, we conclude that

$$|\{c \in B^{\mathfrak{A}}_{6r+3}(\bar{a}) \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}| = |\{e' \in B^{\mathfrak{B}}_{6r+3}(\bar{b}) \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}) \mid e' \sim^r_k e\}|.$$

- By using essentially the same argument we show that

$$|\{c \in A \setminus B^{\mathfrak{A}}_{6r+3}(\bar{a}) \mid c \sim^r_k e\}| \;\theta\; |\{e' \in B \setminus B^{\mathfrak{A}}_{6r+3}(\bar{b}) \mid e' \sim^r_k e\}|.$$

Furthermore, if $|\{e' \in B \setminus B^{\mathfrak{A}}_{6r+3}(\bar{b}) \mid e' \sim^r_k e\}| \geq t$, then by using Lemma 8.5 and the facts that $\mathfrak{A} \equiv^{\mathfrak{F}}_{k'} \mathfrak{B}$ and $N^{\mathfrak{A}}_{3r+1}(\bar{a}) \equiv^{\mathfrak{F}}_{k'} N^{\mathfrak{B}}_{3r+1}(\bar{b})$, we deduce that $|\{c \in A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \mid c \sim^r_k e\}| \geq t$, which is a contradiction. Therefore, $|\{e' \in B \setminus B^{\mathfrak{B}}_{6r+3}(\bar{b}) \mid e' \sim^r_k e\}| < t$, and by using strong uniformity, we conclude that

$$|\{c \in A \setminus B^{\mathfrak{A}}_{6r+3}(\bar{a}) \mid c \sim^r_k e\}| = |\{e' \in B \setminus B^{\mathfrak{A}}_{6r+3}(\bar{b}) \mid e' \sim^r_k e\}|.$$

This concludes the proof of the claim. $\square$

We now continue with the proof of the lemma. Since $\mathfrak{F}$ is strongly uniform, the claim implies that $[c]^e$ is well-defined, and $|[c]^e|\theta|[e]|$ for every $e \in E$.

Define $C = C' \cup f_0^{-1}(D_0)$. We now show that $|C|\theta|D|$, and that there is $f : A \to B$ such that for every $c \in A$, $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b}f(c))$, and $c \in C$ iff $f(c) \in D$. This will prove the lemma.

We first prove that $|C|\theta|D|$. Since $|C_0|\theta|D_0|$, and $\mathfrak{F}$ is strongly uniform, it suffices to show that $|C'|\ \theta\ |(D \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}))|$. This follows from:

(1) $|(D \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b}))| = \sum_{e \in E} |[e]|$;
(2) $|C'| = \sum_{e \in E} |[c]^e|$; and
(3) $|[c]^e|\theta|[e]|$ for every $e \in E$.

It remains to show that there is $f : A \to B$ such that for every $c \in A$, $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b}f(c))$, and $c \in C$ iff $f(c) \in D$.

Choose any function $f_{\text{out}} : A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a}) \to B \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $c \sim^r_k f_{\text{out}}(c)$ for every $c \in \text{dom}(f_{\text{out}})$ and $c \in C$ iff $f_{\text{out}}(c) \in D$. That such an f_{out} exists can be seen from the following:

- If $c \in C'$, then $f_{\text{out}}(c)$ is any element $e \in D \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $c \sim^r_k e$. That such an element exists follows from the definition of C'.
- If $c \in A \setminus B^{\mathfrak{A}}_{2r+1}(\bar{a})$ but $c \notin C'$, we consider two possibilities. Either there is an element $e' \in D \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $e' \sim^r_k c$, or there is no such an element. In the first case there must be an element $e \in B \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b})$ but $e \notin D$ such that $c \sim^r_k e$. Then we set $f_{\text{out}}(c) = e$. In the second case, again we have two possibilities. The first one is that $c \notin B^{\mathfrak{A}}_{6r+3}(\bar{a})$. Since $\mathfrak{A} \equiv^{\mathfrak{F}}_{k'} \mathfrak{B}$ and $N^{\mathfrak{A}}_{6r+3}(\bar{a}) \equiv^{\mathfrak{F}}_{k'} N^{\mathfrak{B}}_{6r+3}(\bar{b})$, from Lemma 8.5 there exists $e \in B \setminus B^{\mathfrak{B}}_{2r+1}(\bar{b})$ such that $c \sim^r_k e$. Since $e \notin D$, we can set $f_{\text{out}}(c) = e$. The second case is that $c \in B^{\mathfrak{A}}_{6r+3}(\bar{a})$. Since $N^{\mathfrak{A}}_{7r+3}(\bar{a}) \equiv^{\mathfrak{F}}_{k+2\lceil\log(6r+3)\rceil+1} N^{\mathfrak{B}}_{7r+3}(\bar{b})$, there exists a function $h : B^{\mathfrak{A}}_{6r+3}(\bar{a}) \to B^{\mathfrak{B}}_{6r+3}(\bar{b})$ such that
$$N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_{k+\lceil\log(6r+3)\rceil} N^{\mathfrak{B}}_r(\bar{b}h(c)).$$

From Lemma 5.8, $h(c) \notin B^{\mathfrak{B}}_{2r+1}(\bar{b})$. Also, from Lemma 5.9, $c \sim^r_k h(c)$. We then let $f_{\text{out}}(c) = h(c)$. Again, $f_{\text{out}}(c) \notin D$, by definition.

Define $f = f_{\text{out}} \cup f_0$. Clearly, f is a function from A to B. It remains to show that $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b}f(c))$ for every $c \in A$ (the fact that $c \in C$ iff $f(c) \in D$ comes directly from the definition of f_0 and f_{out}). This is done by cases:

- For $c \in B^{\mathfrak{A}}_{2r+1}(\bar{a})$, this follows from the definition of f_0.
- For $c \notin B^{\mathfrak{A}}_{2r+1}(\bar{a})$ notice that $N^{\mathfrak{A}}_r(\bar{a}) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b})$, $N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(f_{\text{out}}(c))$, and $B^{\mathfrak{A}}_r(\bar{a}) \cap B^{\mathfrak{A}}_r(c) = B^{\mathfrak{B}}_r(\bar{b}) \cap B^{\mathfrak{B}}_r(f_{\text{out}}(c)) = \emptyset$. Then, from Proposition 5.5 and the fact that $\mathfrak{F}$ is basic, we conclude that $N^{\mathfrak{A}}_r(\bar{a}) \cup N^{\mathfrak{A}}_r(c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b}) \cup N^{\mathfrak{B}}_r(f(c))$. Since $d(\bar{a}, c) > 1$ and $d(\bar{b}, f(c)) > 1$, $N^{\mathfrak{A}}_r(\bar{a}) \cup N^{\mathfrak{A}}_r(c) = N^{\mathfrak{A}}_r(\bar{a}c)$ and $N^{\mathfrak{B}}_r(\bar{b}) \cup N^{\mathfrak{B}}_r(f(c)) = N^{\mathfrak{B}}_r(\bar{b}f(c))$. Therefore, $N^{\mathfrak{A}}_r(\bar{a}c) \equiv^{\mathfrak{F}}_k N^{\mathfrak{B}}_r(\bar{b}f(c))$.

This concludes the proof of the lemma. $\square$

We continue now with the proof of Theorem 8.3. We first show that for every $k, m \geq 0$ there exist $r, s, \ell \geq 0$ such that, for every $\mathfrak{A}$ and $\mathfrak{B}$ over the same vocabulary, and every $\bar{a} \in A^m$ and $\bar{b} \in B^m$,

$$\mathfrak{A} \equiv^{\mathfrak{F}}_{\ell} \mathfrak{B} \quad \text{and} \quad N^{\mathfrak{A}}_r(\bar{a}) \equiv^{\mathfrak{F}}_s N^{\mathfrak{B}}_r(\bar{b}) \quad \Longrightarrow \quad (\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_k (\mathfrak{B}, \bar{b}).$$

This is done by induction on k. For $k = 0$ we simply choose $\ell, r, s = 0$, no matter what m is. For the induction step, we assume that r, s, ℓ witness the statement for k and $m + 1$, and find $r', s', \ell' \geq 0$ that witness it for $k + 1$ and m; that is, for every $\mathfrak{A}$ and $\mathfrak{B}$, and every $\bar{a} \in A^m$ and $\bar{b} \in B^m$,

$$\mathfrak{A} \equiv_{\ell'}^{\mathfrak{F}} \mathfrak{B} \quad \text{and} \quad N_{r'}^{\mathfrak{A}}(\bar{a}) \equiv_{s'}^{\mathfrak{F}} N_{r'}^{\mathfrak{B}}(\bar{b}) \quad \Longrightarrow \quad (\mathfrak{A}, \bar{a}) \equiv_{k+1}^{\mathfrak{F}} (\mathfrak{B}, \bar{b}).$$

From Lemma 8.6 we know that there exist $\ell', r', s' \geq 0$ such that, if

$$\mathfrak{A} \equiv_{l'}^{\mathfrak{F}} \mathfrak{B} \quad \text{and} \quad N_{r'}^{\mathfrak{A}}(\bar{a}) \equiv_{s'}^{\mathfrak{F}} N_{r'}^{\mathfrak{B}}(\bar{b})$$

then there exists $\mathcal{F}(A, B) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(A, B)$ and every $c \in \mathrm{dom}(f)$,

$$\mathfrak{A} \equiv_{l}^{\mathfrak{F}} \mathfrak{B} \ \text{ and } \ N_{r}^{\mathfrak{A}}(\bar{a}c) \equiv_{s}^{\mathfrak{F}} N_{r}^{\mathfrak{B}}(\bar{b} f(c)),$$

and there exists $\mathcal{F}(B, A) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(B, A)$ and every $e \in \mathrm{dom}(f)$,

$$\mathfrak{A} \equiv_{l}^{\mathfrak{F}} \mathfrak{B} \ \text{ and } \ N_{r}^{\mathfrak{A}}(\bar{a} f(e)) \equiv_{s}^{\mathfrak{F}} N_{r}^{\mathfrak{B}}(\bar{b}e).$$

From the latter, we deduce that there exists $\mathcal{F}(A, B) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(A, B)$ and every $c \in \mathrm{dom}(f)$, $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}} (\mathfrak{B}, \bar{b} f(c))$, and there exists $\mathcal{F}(B, A) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(B, A)$ and every $e \in \mathrm{dom}(f)$, $(\mathfrak{A}, \bar{a} f(e)) \equiv_{k}^{\mathfrak{F}} (\mathfrak{B}, \bar{b}e)$. In other words, $(\mathfrak{A}, \bar{a}) \equiv_{k+1}^{\mathfrak{F}} (\mathfrak{B}, \bar{b})$, which completes the proof of the Gaifman-locality of strongly uniform agreements.

It remains to show the second item of the theorem, namely that there exist alternative agreements $\mathfrak{G}(\mathrm{FO})$, $\mathfrak{G}(\mathrm{FO}(\mathbf{Q}_p))$, and $\mathfrak{G}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ which are strongly uniform. They are defined below.

$\mathfrak{G}(\mathrm{FO})$: given $A, B \subset U$, a tactic is a set $\mathcal{G}(A, B)$ of maps such that for every $D \subseteq B$, there exists $g \in \mathcal{G}(A, B)$ such that $\mathrm{dom}(g) = A$ and $g^{-1}(D) = \emptyset$ iff $D = \emptyset$. Then $\mathfrak{G}(\mathrm{FO})$ contains all possible tactics.

$\mathfrak{G}(\mathbf{Q}_p)$: this is just $\mathfrak{F}(\mathbf{Q}_p)$.

$\mathfrak{G}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$: given $A, B \subset U$ such that $|A| = |B|$, a tactic is a set $\mathcal{G}(A, B)$ of maps such that for every $D \subseteq B$, there exists $g \in \mathcal{G}(A, B)$ such that $\mathrm{dom}(g) = A$ and $|g^{-1}(D)| = |D|$. Then $\mathfrak{G}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ contains all possible tactics.

We now show that $\equiv_k^{\mathfrak{F}(\mathcal{L})}$ and $\equiv_k^{\mathfrak{G}(\mathcal{L})}$ are the same, by induction on k, separately for each $\mathcal{L}$. The base case is immediate for all three logics. Now we show how to use the hypothesis for k to prove the statement for $k + 1$.

$\mathfrak{G}(\mathrm{FO})$: It is easy to see that $\equiv_{k+1}^{\mathfrak{G}(\mathrm{FO})}$ is contained in $\equiv_{k+1}^{\mathfrak{F}(\mathrm{FO})}$ (since the duplicator can win the $\mathfrak{F}(\mathrm{FO})$-game by mimicking the strategy of the duplicator in the $\mathfrak{G}(\mathrm{FO})$-game where the spoiler always chooses $D = B$). Thus, we need to prove the converse, namely that $(\mathfrak{A}, \bar{a}) \equiv_{k+1}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b})$ implies $(\mathfrak{A}, \bar{a}) \equiv_{k+1}^{\mathfrak{G}(\mathrm{FO})} (\mathfrak{B}, \bar{b})$. To do this, it is enough to construct a tactic $\mathcal{G}(A, B) \in \mathfrak{G}(\mathrm{FO})$ such that, for every $g \in \mathcal{G}(A, B)$ and every $c \in \mathrm{dom}(g)$, $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}g(c))$ and apply the hypothesis. Consider an arbitrary $D \subseteq B$. Because $(\mathfrak{A}, \bar{a}) \equiv_{k+1}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b})$, for each element $e \in B$, there is $c \in A$ such that $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}e)$, and for each element $c \in A$ there is $e \in B$ such that $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}e)$. Now, for each $e \in B$, let $[e]_A$ be the set of all $c \in A$ such that $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}e)$. Let $C = \bigcup_{e \in D} [e]_A$. Thus, there is a function $h_1 : C \to D$ such that $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}h_1(c))$, and there is a function $h_2 : A \setminus C \to B \setminus D$ such that $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}h_2(c))$ (since for every element $c \in A \setminus C$, there is $e \in B$ such that $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}e)$, and by the definition of C, $e \notin D$). Therefore, $g = h_1 \cup h_2$ is a function from A to B such that for every $c \in A$, $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{F}(\mathrm{FO})} (\mathfrak{B}, \bar{b}g(c))$, and by the induction hypothesis, $(\mathfrak{A}, \bar{a}c) \equiv_{k}^{\mathfrak{G}(\mathrm{FO})} (\mathfrak{B}, \bar{b}g(c))$. Furthermore, since for every $c \in A$, $c \in C$ if and only if $g(c) \in D$, we have $g^{-1}(D) = \emptyset$ iff $D = \emptyset$. This allows us to conclude that $(\mathfrak{A}, \bar{a}) \equiv_{k+1}^{\mathfrak{G}(\mathrm{FO})} (\mathfrak{B}, \bar{b})$.

For $\mathfrak{G}(\mathrm{FO}(\mathbf{Q}_p))$ there is nothing to prove. The case of $\mathfrak{G}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ is very similar to (and in fact simpler than) the case of $\mathfrak{G}(\mathrm{FO})$.

We finally show that $\mathfrak{G}(\mathrm{FO})$, $\mathfrak{G}(\mathrm{FO}(\mathbf{Q}_p))$, and $\mathfrak{G}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt}))$ are basic and strongly uniform. The proof that they are basic is the same as the proof of Proposition 5.2). To show strong uniformity we simply present the equivalence relations witnessing it:

- For $\mathfrak{G}(\mathrm{FO}) - n\theta m$ iff $n = m = 0$ or $n, m > 0$.
- For $\mathfrak{G}(\mathrm{FO}(\mathbf{Q}_p)) - n\theta m$ iff $n - m \equiv 0 \,(\mathrm{mod}\ p)$.
- For $\mathfrak{G}(\mathcal{L}^*_{\infty\omega}(\mathbf{Cnt})) - n\theta m$ iff $n = m$.

This completes the proof of Theorem 8.3. □

Proof of Proposition 8.4. From Theorem 7.1, $\mathfrak{F}$ is Hanf-local. Then for every $k \geq 0$ there is $d, \ell \geq 0$ such that $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{d,\ell} (\mathfrak{B}, \bar{b})$ implies $(\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_{k} (\mathfrak{B}, \bar{b})$. In the following paragraph, we show that for $k' = \ell + \lceil \log d \rceil + \lceil \log (2d + 1) \rceil + 1$, we have that $\mathfrak{A} \equiv^{\mathfrak{F}}_{k'} \mathfrak{B}$ implies $\mathfrak{A} \leftrightarrows^{\mathfrak{F}}_{d,\ell} \mathfrak{B}$. Thus, by Lemma 7.4 we conclude that $\mathfrak{A} \equiv^{\mathfrak{F}}_{k'} \mathfrak{B}$ and $N^{\mathfrak{A}}_{3d+1}(\bar{a}) \equiv^{\mathfrak{F}}_{k'} N^{\mathfrak{B}}_{3d+1}(\bar{b})$ imply $(\mathfrak{A}, \bar{a}) \leftrightarrows^{\mathfrak{F}}_{d,\ell} (\mathfrak{B}, \bar{b})$ and, therefore, $(\mathfrak{A}, \bar{a}) \equiv^{\mathfrak{F}}_{k} (\mathfrak{B}, \bar{b})$ holds since $\mathfrak{F}$ is Hanf-local.

Assume $\mathfrak{A} \equiv^{\mathfrak{F}}_{k'} \mathfrak{B}$. Because $k' \geq 1$, there is $\mathcal{F}(A, B) \in \mathfrak{F}$ such that for every $f \in \mathcal{F}(A, B)$ and $c \in \mathrm{dom}(f)$, $(\mathfrak{A}, c) \equiv^{\mathfrak{F}}_{k'-1} (\mathfrak{B}, f(c))$. Since $\mathfrak{F}$ is a bijective agreement, exactly as in the proof of Lemma 7.4 we conclude that there exists a tactic $\mathcal{G}(A, B) \in \mathfrak{F}$ such that $\bigcup_{g \in \mathcal{G}(A,B)} \mathrm{graph}(g)$ is the graph of a bijection from A to B and for every $g \in \mathcal{G}(A, B)$ and $c \in \mathrm{dom}(g)$, $(\mathfrak{A}, c) \equiv^{\mathfrak{F}}_{k'-1} (\mathfrak{B}, g(c))$. Let h be the bijection whose graph is $\bigcup_{g \in \mathcal{G}(A,B)} \mathrm{graph}(g)$. Because $k' - 1 \geq \ell + \lceil \log d \rceil$, we deduce from Lemma 5.9 that h is a bijection from A to B such that for every $c \in A$, $N^{\mathfrak{A}}_{d}(c) \equiv^{\mathfrak{F}}_{\ell} N^{\mathfrak{B}}_{d}(h(c))$, that is, $\mathfrak{A} \leftrightarrows^{\mathfrak{F}}_{d,\ell} \mathfrak{B}$. This completes the proof. □

Open problem. Can the bound $O(7^k)$ in Theorem 8.3 be lowered?

It is known that Gaifman's theorem, originally proven for FO [8] with the $O(7^k)$ bound (where k is the quantifier rank) can be restated with the $O(4^k)$ bound [15]. If one deals with a weaker notion inspired by Gaifman's theorem that only applies to tuples in the same structure, then the bound can be further lowered to $O(2^k)$. But it is still an open question whether, in the case of game-based locality, or just for the statement of Gaifman's theorem for FO, the bounds $O(7^k)$ and $O(4^k)$ can be replaced by $O(2^k)$.

Acknowledgments

Part of this work was done while the authors were at the University of Toronto, supported in part by grants from NSERC, PREA and CITO. Arenas is also supported by FONDECYT grant 1050701, and Arenas and Barceló by Millennium Nucleus Center for Web Research, Grant P04-067-F, Mideplan, Chile. Libkin is on leave from the University of Toronto, supported by a grant from NSERC, and the European Commission Marie Curie Excellence grant MEXC-CT-2005-024502.

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