An introduction to the Probabilistic Method

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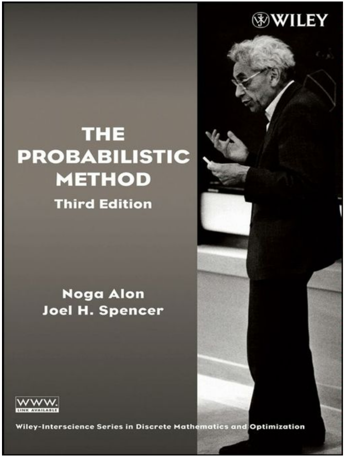
The first words about induction

"Even though this proposition may have an infinite number of cases, I shall give a very short proof of it assuming two lemmas. The first, which is self evident, is that the proposition is valid for the second row. The second is that if the proposition is valid for any row then it must necessarily be valid for the following row."

Blaise Pascal, 17^{th}*century*.

Standard references



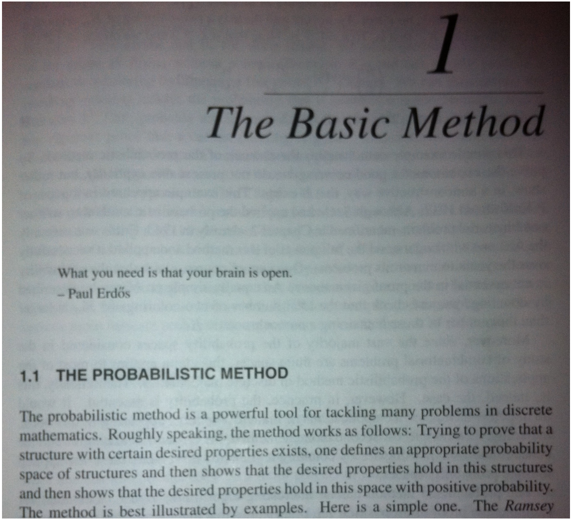
And the first pages...

LECTURE 1

The Probabilistic Method

Ramsey $R(k, k)$. The probabilistic method is best described by examples. Let us plunge right in. Let $R(k, t)$ denote the Ramsey function, i.e., the minimal n so that if the edges of K_n are two-colored Red and Blue then either there is a Red K_k or a Blue K_t. That $R(k, t)$ is well defined, i.e., its holding, for n sufficiently large, is Ramsey's Theorem, which will not be our concern here. Rather, we shall examine *lower bounds* to the Ramsey function. By simple logic:

And the first pages...



1

The Basic Method

What you need is that your brain is open.
– Paul Erdős

1.1 THE PROBABILISTIC METHOD

The probabilistic method is a powerful tool for tackling many problems in discrete mathematics. Roughly speaking, the method works as follows: Trying to prove that a structure with certain desired properties exists, one defines an appropriate probability space of structures and then shows that the desired properties hold in this structures and then shows that the desired properties hold in this space with positive probability. The method is best illustrated by examples. Here is a simple one. The *Ramsey*

The first example

Definition

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(Natural) question:

Given any k, how big is $R(k)$?

The Ramsey Numbers

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What if such an upper bound is less than 1?

The Ramsey Numbers (cont.)

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Given any k-subset R of K_n, let A_R be the event of R being *monochromatic*.

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$$P[A_R] = 2^{1-\binom{k}{2}}.$$

Then the probability that at least one A_R occurs is at most $\binom{n}{k}2^{1-\binom{k}{2}}$.

The Ramsey Numbers (cont.)

Corollary

If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ *then* $R(k) > n$.

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Corollary

If $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ *then* $R(k) > n$.

Let $f(k, n) = \binom{n}{k}2^{1-\binom{k}{2}}$.
Here are some sample values:

k	n	$\approx f(n, k)$
4	6	0.46875
6	17	0.75537
10	100	0.98397
18	2639	0.99488
34	1225000	0.89015

A little look back

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3) Choose a random structure in the space.
4) Prove that the property holds with strictly positive probability.

Tournaments

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A tournament is a directed graph $G = (V, E)$ such that for every $x, y \in V$ with $x \neq y$, either $(x, y) \in E$ or $(y, x) \in E$ but not both.

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We say a tournament $G = (V, E)$ has the property S_k if for every k-subset A of V exists a vertex $v_A \notin A$ such that for every vertex $x \in A$, $(v_A, x) \in E$.

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Any existencial questions?

The property S_k

Given a positive integer k we want to find n such that there is a tournament with n vertices satisfying S_k.

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- Find an upper bound $M = f(n,k)$ for the probability of randomly choose a tournament with n nodes without the property S_k.

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How to proceed:

- Find an upper bound $M = f(n, k)$ for the probability of randomly choose a tournament with n nodes without the property S_k.
- Find restrictions for n and k such that $M < 1$.

The property S_k (cont.)

Consider a random tournament $T = (V, E)$ with n nodes. For every k-subset R of V, let A_R be the event that there is no vertex in $V \setminus R$ that beats every node on R.

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$$P\left[\bigcup_{\substack{R \subset V \\ |R| = k}} A_R\right] \leq \binom{n}{k}(1 - 2^{-k})^{n-k}$$

The property S_k (cont.)

Corollary

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Let $g(k,n) = \binom{n}{k}(1-2^{-k})^{n-k}$.
Some sample values:

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How can we find a tournament satisfying S_4?

The algorithmic point of view

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$$\text{What if } g(n,k) \approx 0?$$

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5	1200	0.00068
6	5000	$1{,}5 \cdot 10^{-15}$

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$$\text{What if } f(n,k) \approx 0?$$

Then it is very likely to randomly find a two-coloring of the edges of K_n with no monochromatic K_k.

The algorithmic point of view (cont.)

Surprisingly, for not so large k, there are large n for which it is very likely to find such colorings:

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6	17	0.75537
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k	n	$\approx f(n,k)$
4	5	0.15625
6	10	0.01281
10	50	0.00058
18	1500	0.00003
34	600000	$2{,}5 \cdot 10^{-11}$

Expected value

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Let X be a random variable that can take value x_1 with probability p_1, x_2 with probability p_2, ... , and x_n with probability p_n.Then the expected value of X is

$$E[X] = \sum_{i=1}^{n} x_i \cdot p_i$$

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Example

Let σ be a permutation on the set $\{1, 2, \ldots, n\}$.
What is the expected value for

$$X = |\{a \in \{1, \ldots, n\} \mid \sigma(a) = a\}|?$$

Expected value applications

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Here is an example:

Theorem

Let $G = (V, E)$ be a graph with e edges. Then G contains a bipartite subgraph with at least $e/2$ edges.

Splitting graphs

Proof

Let $R \subseteq V$ *where for every node* x *in* V, $P[x \in R] = 1/2$, *this choices being mutually independent.*

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Let $X_{\{x,y\}}$ *be the indicator variable for* $|\{x, y\} \cap R| = 1$.

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$$X = \sum_{\{x,y\} \in E} X_{\{x,y\}}$$

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$$X = \sum_{\{x,y\} \in E} X_{\{x,y\}} \Rightarrow E[X] = \sum_{\{x,y\} \in E} E[X_{\{x,y\}}]$$

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As $E[X_{\{x,y\}}] = 1/2$ *for every* $\{x, y\} \in E$, *we conclude there is a bipartite subgraph with at least* $e/2$ *edges.*

□

The first application

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For every n, there is a tournament T with n players and at least $n!2^{1-n}$ Hamiltonian paths.

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For every n, there is a tournament T with n players and at least $n!2^{1-n}$ Hamiltonian paths.

Here we say there exists a tournament, so it will be randomly chosen.

The first application (cont.)

Proof

Let $T = (V, E)$ be a tournament with n nodes. Let X be the number of Hamiltonian paths on T.

The first application (cont.)

Proof

Let $T = (V, E)$ be a tournament with n nodes. Let X be the number of Hamiltonian paths on T.
For every strict total order $<$ of the n nodes, let $X_<$ be the indicator random variable for $<$ giving an ordered hamiltonian path.

The first application (cont.)

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Let $T = (V, E)$ be a tournament with n nodes. Let X be the number of Hamiltonian paths on T.
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$$E[X] = \sum_{< \text{ order for } V} E[X_<]$$

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$$E[X] = \sum_{< \text{ order for } V} E[X_<]$$

As $E[X_<] = 2^{1-n}$ for every linear order, and there are $n!$ orders for the n nodes, we conclude there is a tournament with $n!2^{1-n}$ Hamiltonian paths.

$\square$

A lower bound for independent sets

The following theorem was proved independently by Caro (1979) and Wei (1981).

Theorem (Caro and Wei)

Let $G = (V, E)$ be a graph with n vertices. Then

$$\alpha(G) \geq \sum_{v \in V} \frac{1}{deg(v) + 1}$$

A lower bound for independent sets (cont.)

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$$I_< = \{v \in V \mid \{v, w\} \in E \Rightarrow v < w\}.$$

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Let $X_v = \mathbb{1}_{v \in I_<}$. As $v \in I_<$ iff it is the least element among v and its neighbors, $I_<$ is independent

A lower bound for independent sets (cont.)

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A lower bound for independent sets (cont.)

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Let $<$ *be an uniformly chosen order for* V. *Define*

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Let $X_v = \mathbb{1}_{v \in I_<}$. *As* $v \in I_<$ *iff it is the least element among* v *and its neighbors,* $I_<$ *is independent and* $E[X_v] = \frac{1}{deg(v)+1}$.
Let $X_< = |I_<| = \sum_{v \in V} X_v$.

$$E[X_<] = \sum_{v \in V} E[X_v] = \sum_{v \in V} \frac{1}{deg(v) + 1}.$$

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Proof

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Let $X_v = \mathbb{1}_{v \in I_<}$. As $v \in I_<$ iff it is the least element among v and its neighbors, $I_<$ is independent and $E[X_v] = \frac{1}{deg(v)+1}$.
Let $X_< = |I_<| = \sum_{v \in V} X_v$.

$$E[X_<] = \sum_{v \in V} E[X_v] = \sum_{v \in V} \frac{1}{deg(v)+1}.$$

Thus, for some order $<$, $I_<$ has at least $\sum_{v \in V} \frac{1}{deg(v)+1}$ nodes. ❑

Crossing Numbers

Definition

An embedding of a graph $G = (V, E)$ on a surface Σ is a pair (f, g) where $f : V \to \Sigma$, $g : E \to A(\Sigma)$ (where $A(\Sigma)$ is the set of simple arcs on Σ) such that for every $\{x, y\} \in E$, $g(\{x, y\})$ starts in x and ends at y.

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Definition

The crossing number of a planar representation is the number of pairs of intersecting curves on $g(E)$ with no common endpoints. The crossing number of a graph G, $cr(G)$, is the minimum possible crossing number in a planar representation of G. If $cr(G) = 0$, G is said to be planar.

Crossing Numbers (cont.)

Lemma

The crossing number of any simple graph $G = (V, E)$ is at least $|E| - 3|V|$.

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Proof

Euler's formula: If $G = (V, E)$ is a simple connected planar graph then $|V| - |E| + f = 2$ where f is the number of divisions of the plane in any planar representation of G.

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Euler's formula: If $G = (V, E)$ is a simple connected planar graph then $|V| - |E| + f = 2$ where f is the number of divisions of the plane in any planar representation of G. It follows that $|E| \leq 3|V| - 6$, so $|E| < 3|V|$.

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□

Crossing Numbers (cont.)

Theorem (Ajtai, Chvátal, Newborn and Szemerédi (1982))

The crossing number of any simple graph $G = (V, E)$ with $|E| \geq 4|V|$ is at least $|E|^3/64|V|^2$.

Crossing Numbers (cont.)

Proof

Let $G = (V, E)$ *be a graph with* $|E| \geq 4|V|$ *and* (f, g) *a planar representation of* G *with crossing number* t*. Let* H *be an induced subgraph obtained by choosing randomly and independently each vertex of* G *with probability* p*.*

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Crossing Numbers (cont.)

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By the above lemma $p^4 t \geq p^2|E| - 3p|V|$, so

$$t \geq \frac{|E|}{p^2} - \frac{3|V|}{p^3}$$

Crossing Numbers (cont.)

Proof

Let $G = (V, E)$ be a graph with $|E| \geq 4|V|$ and (f, g) a planar representation of G with crossing number t. Let H be an induced subgraph obtained by choosing randomly and independently each vertex of G with probability p. The expected number of vertices is $p|V|$, the expected number of edges is $p^2|E|$ and expected number of crossing edges in the planar representation of H obtained by the restriction of (f, g) to H is $p^4 t$.
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Now substitute $p = 4|V|/|E|$ (≤ 1) to obtain the desired result. ❑

¿Questions?

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